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MATHEMATICS AND REALITY

(A classic view)

Oliver E. Glenn

Mathematical Introduction

Space and time have both been reasoned, perhaps the most satisfactorily, by Immanuel Kant. He designated them as merely the conditions for experience with objects of sense. They are fully represented a priori in the mind; but, space affords us no cognitions* (1) (2). Space contains nothing movable. We are sure intuitionally only of the fact that space has extensions. In this article we show that an induction based on extension leads to a new ordering of realities.

The only synthetic axiom of elementary euclidian geometry: The straight line is the shortest distace between two points in space, requires that thought should rise from the idea of distance to that of straight (line), not contained in distance. In this axiom, two spacial extensions (of zero extent), come under relationship by means of a unique third spacial extension (of non-zero extent), and the latter is judged* to have, a priori, the character of an object, above mere extent. A proof that the straight (unique) line is the shortest, establishes a synthesis, replaces the intuitive judgment by a concept, and identifies a law of nature. A proof has been furnished by the calculus of variations. (3)

It is, however, significant that this synthesis has generalizations. The following is like one due to Fréchet: Suppose S to be a closed, surface-like space-extension, like a sphere but, for greater generality, having mild undulations. The totality T of spatial extensions, like closed curves, which can be imagined on S is sometimes called a space, but, preferably, a spatial set. A spatial set is not objective. It is only a play on the unrepresented concept of extension. We then choose any two space-extension curves E, F of T, place a point e on E and a point f on F, assume that e and f have the ability of locomotion, each around its closed curve, and let L be the geodesic on S from e to f. As e, f move, the geodesic will have to contract or stretch, that is, its space-extension is variable. In these circumstances there is a minimum value of L, say L_m, idem est, our concept is raised to the

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*(See references at the end of the article).

(1) (K, p.139) "In a cognition there are two elements: first, the conception whereby an object is cogitated (the category); and, secondly, the intuition, whereby the object is given", (represented).

(2) (R, p.27) "Weil wir eine unmittelbare Kenntnis des wirklichen Raumes nicht haben".

(3) Cf. (D_2). *In an affirmative judgment either the predicate B belongs to the subject A as somewhat that is contained in A (though covertly); or B lies completely of of A, although standing in some connection with it. In the first case the judgment is said to be analytic; in the second it is said to be synthetic.

intuition of a unique geodesic of the system of geodesics. Thus E, F come under a definite relationship by means of a unique line L_m. Since uniqueness is not contained in relation, we have reached a synthetic judgment. Proof of the minimum property will change this judgment into a synthesis and establish a law of nature, the intuition concerning L_m becoming a concept.

The straight line previously mentioned, and the minimal L, are then objective realities. (4) They are not spatial realities, primarily, though they have extension.

Heuristic identification of an ascending scale of realities.

These principles of mathematical reasoning are obviously considerably broadened in the following generalization which leads to a hierarchy Σ of realities. We deal with a reality, of course, not as a thing in itself, but as a representation of a concept in a cognition. Synthetic judgment is the basis of the structure of the scale Σ of realities. If A is any reality (representation), there may be a unique property B which, as a reality, is not in A but is found, in the intuition, as belonging to a reality different from A. But since the judgment that A has B as a property, is synthetic, we rule that the second reality ranks above A in the scale.

For example, the pair of space-extension curves (E, F), considered above, is a spatial reality A. The line L_m is uniquely related to (E, F) but is not of the same reality as (E, F). The line L_m is an objective reality, and the predicted proof, that it is uniquely related to the spatial reality, will establish a synthesis.

On the basis of the rule mentioned, we have constructed the following:

Scale Σ of realities.

($\varkappa$) Life-perceptual cognitions; (ι) Creative cognitions (spontaneous mentality); (θ) Libertarian (Cognitions of the true, beautiful, useful, good); (N) Racial (of the humane); (ζ) Sensory (response of the wave-intuition to the a priori concept of an experience); (ε) Tribal-emotional (subliminal mentality); (δ) Subsistential (of actual and intuited self-preservation); (γ) Phenomenal (of occurrence); (β) Existential (of objectivity); (α) Spatial-temporal.

Since we pass from each degree to the next higher by an intuition that gives rise to a synthetic judgment, we can also attach an index (0 to 9), to each degree Σ, to show its order of synthesis. A degree can contain any reality lower than itself, in Σ. The degree is limited above by its synthesis of highest index. The human organism is a structure of realities of all ten degrees. The scale definitely restricts our linguistic usage, in the case of the words object, phenomena, and others.

(4) (K, p.162) "Reality, in the pure conception of the understanding, is that which corresponds to sensation in general; that, consequently, the conception of which indicates a being (in time)".

The specified connection between the sensuous representation in a degree of index i of Σ, and the representation in the related degree of index $i + 1$, is not relative to the concept in the cognition (i), but only to the representation in this cognition, because the connection is recognized by the intuition only. In this sense the connection exists in opposition to the concept or, as Hegel implies, is an antithesis and an origin of the synthesis.

(γ) *The phenomenal realities.* An object in (β), as a stone A, will be uniquely related to a property (as its weight B), that is not a reality of (β) but is found in the intuition as a phenomenon, idem est, a reality of (γ). Hence to say that A has B is a synthetic judgment, uniqueness being not contained in relation. The jûdgment verifies the existence of the reality (γ) and gives it precedence over (β) in Σ. Proof of the stated property of uniqeness furnishes a synthesis, a concept, and a law of nature. The objective degree of realities, as exemplified by the stone, may be said to contain the possibility of the connection implied by weight. (5)

(δ) *The subsistential degree.* A phenomenon in (γ) may contain the property of becoming self-perpetuating. It seems that a virus is an example of this.** It is a phenomenon (γ) and it is related to this property of subsistence, which is not in (γ) but is found in the intuition and judged to be something which belongs to the degree (δ) of subsistential realities. Another example of δ is a sheep's grazing activity, or a squirrel's burying nuts in the ground for winter use. In any such representation the connection with (δ) is judged to be unique. With proof, the judgment, that A has B, becomes a synthesis, leads to a concept, and places (δ) above (γ) in Σ.

(ε) *The degree of subliminal mentality.* Subsistence as a reality leads us to the intuition that the subsistence of the organism is connected with an element of mental control, but this mentality would not be in (δ). It would be found in the intuition as a reality in (ε), the degree of subliminal mentality. Proof of the uniqueness of its connection would furnish the necessary synthesis (δ, ε), and a law of nature.

(ζ) *The sensory degree.* When a man studies a Greek vase, he may connect with his sense of form, which is a priori and aesthetical. An animal may have a sense of pity. We readily intuit that any organism possessing a subliminal mentality in (ε) will find it uniquely related to some property of sense. But this property, as a sensory reality, is not in (ε) but in (ζ), the degree of "a sense of". Again, proof of

(5) (H, p.108) "So, reflectively at least, possibility points to something destined to become actual".

(K, p.171) "The possibility of experience, then, is that which gives objective reality to all our a priori cognitions".

** H. F. Osborn, *The Origin and Evolution of Life* (1916).

the uniqueness of the intuited relation, will furnish a synthesis (ε, ζ) and give (ζ) precedence over (ε) in Σ.

(N) *The degree of the humane.* We readily intuit that an organism which possesses a sensory reality (ζ), will contain an urge to think and do something about it. But this urge is not sensory. It is found in the intuition as something belonging to the degree (N) of the realities of humane thought and action. But there will be the possibility of proof that the connection between the aforesaid reality in (ζ) and the urge in (N), is unique, which proof would establish the necessary synthesis and fix the position of (N) in Σ, synthesis because, as before, uniqueness is not contained in relation.

(θ) *The libertarian degree.* When we consider an organism possessing a specific racial or humane reality (N), we readily intuit that this reality implies also the presence and connection of a related moral principle. But this moral reality is not of the humane degree (N). It is found in the intuition as something which appertains to the degree (θ) of libertarian realities. It is conceivable, also, that the uniqueness of its connection could be proved, to establish a synthetic concept, and a law of nature.

(ι) *The spontaneous or creative degree.* If a man has a moral principle, idem est, possesses a reality of the degree (θ) of the true, beautiful, useful, good, the inference is that he will have, also, a related desire to improve or enhance this principle, which is the meaning of the spirit of research. But this desire is not of the libertarian degree of realities, a fact that finds abundant verification in history. We intuit and judge this desire as being a reality of the degree (ι) of creative spontaneity. Proof of the uniqueness of its connection would not be easy, but, when accomplished, it will establish the synthesis (θ, ι) that the intuition implies. We see that the problem of proof can at least be stated, which is all that can be claimed for the proofs mentioned from (γ) to $(\varkappa)$.

$(\varkappa)$ *The degree of the realities of life-perception.* Any function of life, in particular a creative mental function, will evidently be a subordinate reality in comparison with some realities under the aegis of a complete knowledge of the nature and processes of life. But we can intuit and judge that a specific creative function (ι), as research in cybernetics, will contain some reality that will be in unique connection with a reality in $(\varkappa)$. A conceivable proof of the uniqueness of this relation would establish the synthesis $(\iota, \varkappa)$ which is implied by this intuition. Of course, a complete knowledge of the nature and processes of life has not been attained.

However our scale Σ, is very well established on the side of intuition, which, as we have introduced Σ, does not lead us beyond

possible experience. (6)

Earlier advances and predictions.

Croce stated that reality is history and is only historically known.(7) He meant, in particular, that one writing the description of an "occasion", and giving due attention to its contemporary bearings, has as his sole resource, the known of history, which should present what is relevant. But it is also certainly true that this principle should also be clear from the point of view of science, idem est, of mathematics. The scale Σ is a means for such clarification, for the process of evolution surely began to operate when the degree (δ) came into being. Some form of the process may have been implicit in connection with even lower degrees, as the above geometric considerations may suggest. The scale has been built, in time, by the selection program of evolution, operating in harmony with the principle of synthesis, and this identifies Σ with history.

We assume that all realities are represented within Σ. Other intuitions will be found to be outside of the limits of cognitions of reality.

However, something further sould be stated about synthetic judgment. If A predicates B, the judgment is synthetic or analytic according as B is without or within A, but suppose that B is neither without nor within A, but is on the border of A.(8) A synthetic judgment leads to a superior reality, as we have seen. An analytic judgment leads again to a reality, in A or in a lower degree or in both. However, consider the judgment: An organism has the subliminal mentality of ferocity. This can be a subliminal mental trait connected with subsistence (δ) but found also in (ε), or it can be a subliminal mental trait connected with the sense of prowess (ζ). We can only conclude that B is on the border of A But in such a case, as in all other cases, the theorem holds that Σ is invariant under all types of judgment.

Hegel was the thinker, of the nineteenth century, who came closest to the idea of a scale of realities. He had the concept but did not represent it in any very empirical form. He regarded history (of the world) as the progressive realization of the idea of freedom, human history, reason, and knowledge of truth being regarded by him as the highest level in a gradation that rises from inorganic nature

(6) (K, p.151) "No a priori cognition is possible for us, except of objects of possible experience".

(7) (C, p.65). (Croce, *Aesthetic*, p.30): "The world of what has happened, of the concrete, of historical fact, is the world called real, natural, including in this definition both the reality called physical and that called spiritual and human".

(8) (R, p.29) "Diese Stube zerlegt die Stellen, an denen ich mich befinden kann in drei Klassen, in solche innerhalb der Stube, in solche ausserhalb der Stube, und in solche welche weder im Inneren noch im Äusseren der Stube sind".

to genius. He acknowledged no other cause back of the progress than thought acting to integrate a thesis, and its antithesis, into a synthesis. This referred to the influence of negative characteristics upon the positive, in a nation, but, theoretically, more especially to the opposition that we identified above as existing between the intuitive connection and the concept, in a cognition, when there is a synthetic judgment. Hegel regarded evolution as a type of necessity, or at least as a logical process. His world was a creation by synthesis, and that is about what we have said about the scale of realities Σ, which is the empirical result of synthesis, although the series of proofs mentioned as outstanding may yet require a long time. Note that degree (α) cannot be entered, nor ($\varkappa$) deserted, by means of a synthesis.

Giambattista Vico (*Scienza Nuova*, 1725), identified Σ from ζ to θ, in relation to his theory of the nature of poetry. He said that poetry precedes intellect, but follows sense, idem est, poetry is primarily a humane reality (N). The human mind "makes use of intellect when from things which it feels by sense it gathers something that does not fall under sense".

Kant did not reach the idea of a scale, although he was near to it in a few lines of the *Kritik der reinen Vernunft*.(9) He approached the realities so to say, from the top of the scale, by presenting the theory of the a priori in cognition, under which theory an outward reality is not cognized as a thing in itself but as "a mere representation of our sensibility".

Undoubtedly the mind has developed the capacities of the a priori on a program of its own during the history of evolution. We would call the a priori a synthesis from the mental experiences of the race. Since all elementary experiences have had their influences upon this synthesis, the hierarchy of realities Σ will permit us to define, with Kant, the realities of the lower degrees, at least, as abstractions, that is, intuitive representations in cognitions, because a priori intuitions of the inferior realities have been put in the mind during the course of evolution.

A basis for mathematical criticism.

Mathematical criticism, as far as it may be said to exist at all, in print, is far from adequate, in view of the large number of articles of importance which are constantly appearing. We might say, incidentally, that the literary quality of a piece of mathematical writing will justly be a subject for criticism. Any informational subject can be very interesting if the writing is interesting in the literary sense.

(9) (K, p.51) "Thus it is experience upon which rests the possibility of the synthesis of the predicate of weight with the conception of body, because both conceptions, although the one is not contained in the other, still belong to one another (contingently)".

But, more fundamentally, it has not been clear enough as to what kinds of realities the mathematician is supposed to be dealing with.

Use of the term mathematical reality implies a representation, by the intuition, of a mathematical concept; however, representation in what degree of reality? A proof of a theorem results logically from a complex of concepts, among which will be fundamental axioms, but proof does not invariably do more than present the reality, expressed by the theorem, in the intuition, the reality being often, or usually, in the degree (θ). The reality will be intuitional still, and not conceptual, as long as we have to realize that the postulates may eventually have to be revised by some one. In the realm within which the postulates remain unquestioned, the proof replaces an intuition by a concept. Often it is possible to go farther and secure a representation in another degree. In a verification of the theorem by physical experimentation, realities of the phenomenal degree (γ) will play their part, or, farther down the scale, the theorem may furnish objective representations in (β), that is, geometric lemmas. Geometric lines are objects (β), in fact, Clifford has written: "Geometry is a physical science".

Number is a synthetical language that we use to find our way among spatial and temporal realities. Any science, to be satisfactorily complete, should be able to orient itself in all degrees of reality Σ. This rule is the main principle upon which depends the unity of knowledge.

Mathematics is the science that has been the most completely extended into the various degrees of Σ, although there is still some incompleteness. This fact has been but imperfectly understood. Modern mathematics treats of spatial and temporal realities (α) through their objective syntheses (β). It treats extensively of phenomena (γ) in physics, chemistry, mathematical mechanics and the like. It treats of the subsistential (δ), and of subliminal mental realities (ε), somewhat, in bio-mathematics. An intriguing branch of mathematics, which relates to the sense of harmony, (ζ), dates all the way from Pythagoras, and there is mathematics of the sense of form, in the case of plastic art. The comparatively recent mathematics of economics deals with the humanistic realities, as does, also, the mathematics of games.

We note that, in any case where mathematics is applied to a reality within a degree, as in the mechanics of central orbits, within (γ), synthesis can be effected in two ways. Either the mathematics, or the reality beneath the mathematics, can be synthesized. In the former case, for example when Bode's law, as it is exemplified in our solar system, is generalized to an arbitrary planetary system, synthesis leaves the theory within the same degree (γ). In the latter case, synthesis, properly effected, leads to a mathematical theory in the next higher degree, here (δ).

In (δ) we are concerned with biological subsistence. A comprehensive

theory relating to it is that of the curves obtained by plotting the magnitude r of a biological character, as the maximum lifting force of a biceps muscle, or the skull capacity, by means of the angle θ representing the time. Since r is near zero in the embryo and increases with the time, the curve is a spiral, and this spiral, drawn for each of the specimens in a line of heredity, gives a relevant field F of spirals, infinitesimally spaced. Then a synthesis is possible from the phenomenon of obbital distribution in (γ) to the reality of subsistence in (δ).

What is the appropriate description of this synthesis? Since the mathematics is not here being synthesized, the spirals chosen in (δ) to exemplify the theory, must also be stable orbits in (γ). Spirals of Archimedes can be chosen, with equation $r = b + a(\theta - \pi)$. For if we substitute from this equation, in the equation for orbits,

$$(d^2u)/(d\theta^2) + u = P/h^2 u^2, \quad (u = 1/r),$$

the force is found to be,

$$P = h^2(2a^2 + r^2)/r^5,$$

and this formula is a particular case of the known force for stable orbits in general.

The rotating planet in (γ) synthesizes to the phenomenon of growth in (δ), The phenomenon of rotation in (γ) synthesizes to subsistence in (δ), which reality will be possible only if growth conforms to the geometric field.

Some planetary orbits maintain a static position and restore this position when they are perturbed by any small outside attracting force. Correspondingly, in the synthesis, some organisms maintain a static field F from which their curves vary only by negligibly small amounts, and are accordingly said to have become stabilized with respect to the character that determines F. Again, some planetary orbits illustrate the phenomenon of an advancing perihelion, and correspondingly, in the synthesis, the curves of the field F, considered chronologically, may continually expand, any spiral falling outside of its predecessor. In the species the character is then increasing. Advance of perihelion being a phenomenon in (γ), the relevant reality in (δ) is subsistential.

Since the force function, for example P, is a part of the mathematics, it is not synthesized; however, some interpretations are desirable. Since the solar system as a whole is moving in space, an orbital circuit is never repeated as to its position in space, and each circuit may be said to have its accompanying central force (P). In the synthesis, the force of growth, for any spiral, is centered in the accompanying organism which may be arbitrarily placed at the center of the spiral. The respective organisms corresponding to F also change positions but may be assumed to be arranged in positions to correspond

to those of the orbital centers of force in space. Thus there is clearly no synthesis with respect to the forces. We do not really know what either force is as a thing in itself although the physical nature of both may yet be discovered.

Summarizing, we say that the mathematics of a reality in a degree of Σ can be synthesized horizontally leaving the theory within the same degree of Σ, or the reality under the mathematics can be synthesized vertically in Σ. This will give a mathematical theory of a reality in the next higher degree.

Criticism has to take account of various types of errors. These include logical errors, which take in errors in the use of whatever manipulative calculus is being employed. They include the error of falling short of legitimate objectives, and they include the jeopardizing of the intuition because of an imperfect knowledge of the conceptual postulates. This latter type of error is closely related to what may follow from an imperfect estimate on the relation of the mathematics being created to the realities as represented by the scale Σ.

We shall mention one instance of the latter type of hazard. A proof by Dedekind (Cf. D_1), based on the hypothesis that space can be represented by a point-set, shows that, although a region of space may be filled by an everywhere compact set, yet, it will be possible to draw a line, through the region, which passes through no points of the filling.

I remember that Professor Schwatt, in his lectures on the theory of functions of a real variable, expressed dissatisfaction with this result. He seemed to regard it as an antinomy.

But, although a boy walking on stilts may be said to be obtaining a thorough knowledge of the region through which he is passing, he is not securing a thorough knowledge of the ground. A point-set, even though it may be everywhere compact, is still an objective reality in (β), whereas space is made up of realities in (α), below in Σ. This makes it seem more reasonable that a proof relating to space, based altogether on objective realities in (β), may be subject to the type of error here under consideration, that of misinterpretation of postulational concepts. A like hazard would be encountered if one should attempt to prove some property, of an objectivity in (β), alone from phenomena in (γ) connected with that objectivity. Incidentally the type of error here referred to even contradicts Kant's principle that space affords us no cognitions. (Compare also R, p.12).

(over)

REFERENCES AND BRIEF SUPPLEMENTARY DEFINITIONS

Kant, *Kritik der Reinen Vernunft* (1787), (Transl. by Meiklejohn; referred to as (K)).

Reidemeister, *Geist und Wirklichkeit* (1953); (R).

Croce, *History as the Story of Liberty* (1941), (Tr. from the Ital. by Sprigge; (C)).

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Dedekind, *Stetigkeit und Irrationale Zahlen*; (D_1).

Dienger, *Grundrisz der Variationsrechnung*; (D_2).

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Antithesis. Anything contradictory toward a positive thesis or regime.

A priori. Presented by the inner consciousness acting independently.

Concept. A generalized idea of the essence of the realities included under the idea.

Empirical. Referring primarily to experience.

Geodesic. Shortest distance between two points on a surface.

Heuristic. Aiding or inciting toward discovery.

Intuition. Capture of truth (tentative) by internal apprehension.

Libertarian. Liberty is the moral ideal and its invariant elements are the true, beautiful, useful, and good, (Croce).

Objective realities. Corporeal things. Example: rocks.

Phenomenon. Something occurring.

Subliminal. Subconscious as far as the idea of free choice is concerned.

Subsistential realities. Whatever enables self-perpetuation.

Spatial realities. Examples are various space-extensions considered conceptually only. Their representations are null.

Synthesis. Combination of separate elements into a whole, by a process that imparts new properties to the product.

O. E. Glenn, Lansdowne, Pa.

MATHEMATICS AND REALITY

(A modern view)

Hugh Miller

Questions concerning its relation to reality may be raised with respect to any science. Does physical science really disclose a physical structure out there in universal nature; or is physical theory just a mental construct that facilitates, we know not why, fairly reliable prediction? We should see that such questions unconsciously appeal to a knowledge superior to and independent of the science that is critically assessed. Indeed, only certain knowledge could provide criteria revealing other knowledge to be less than certain.

Today, when most of us are empiricists who take natural science to deliver at best a probable and approximate knowledge, such as is gained by induction from a limited survey of observable occurrence, the apparent self-evidence of its generative principles leaves mathematical science peculiarly vulnerable to skeptical investigation. We cannot very well aver that two and two are only probably and approximately four. What are we to make of these numerical certainties? Must we in this empirical age still make room for absolute number knowledge? Or should we deny to mathematics, solely on the ground of its self-evident certainty, the status of descriptive truth, and equate authentic knowledge with the special sciences – the so-called "descriptive sciences" – that deliver only probable and approximate knowledge?

Thinkers have divided upon this issue. From Plato through Descartes to Bertrand Russell, rationalistic philosophers have held mathematics to furnish authentic truth, disclosing a mathematical necessity imposed upon whatever exists, and providing an indispensable foundation for more special and less certain knowledge (for how, without mathematics, should we assess what is more or less probable?) Yet why should a reason so infallibly endowed be limited to mathematical truth? Until not so long ago the inclusion of geometry within mathematical science required the postulate of geometric necessity; and it was impossible clearly to distinguish geometric from mechanical or physical necessity, or this from chemical, biochemical, physiological, psychological, and other necessity. The rationalist could not call a halt to reason; he was obliged to extend the domain of absolute rational knowledge until it covered the whole of theoretical knowledge. And this was a *reductio ad absurdum* of the rationalistic thesis, for it is evident that physical and other science dependent on induction is less than absolute.

So rationalism always generated its antithesis in an empirical

philosophy of science. The empiricist bluntly equates authentic knowledge with probable knowledge reached by induction. He must accordingly explain mathematical science away, deny to it all cognitive status, refuse it a place within descriptive science and natural knowledge. This empirical criticism has reached its sharpest expressions in the positivism or logical empiricism of today. Mathematics, argues the contemporary positivist, is properly a part of logic; and what logic discloses is a set of rules or prescriptions guiding symbolic construction and securing intelligible expression. Adherence to these rules gives to all explicit description a syntactical structure; but we must not mistakenly suppose that this syntax describes anything in nature (that is to say, outside of language). Symbolic structure is something specifically human. So what the rationalist mistook for a universal necessity imposed upon everything that exists turns out to be only a specific condition of human intercommunication. In reality there is neither mathematical nor other necessity, the only "necessity" known to us being the logical complusion we exert upon ourselves in the interests of intelligible communication. It follows that nature, which knows no necessity, is the realm of the contingent or possible; and it is these possibilities of natural occurrence that we discover, and as probabilities assess, in descriptive science.

Each of these antithetical philosophies, we shall see, has its partial insight; yet both are false, and how shall we avoid the one without falling into the other? Our sole refuge from the philosophical misinterpretation of science lies in science itself. What, therefore, *for the scientist*, is the reality engaged by natural knowledge? It is, of course, whatever the scientist finds it to be. The physicist discovers energies, electrons, light-waves, quanta, etc.; the chemist finds atoms, ions, molecules, crystals; the biologist learns of genes, cells, organisms, species; and so on. But does not science discover any pervasive features attributable to nature at large? May not science have its own metaphysics, like to none excogitated by the philosopher? Well, the scientist does take nature to be everywhere characterized by particular differences, with the different particulars usually exhibiting specific similarities. This much might be said. It seems to be all that can be universally said.

For science, as for common sense and presumably for animal intelligence, there is indubitably a *plurality of particulars*. There are many particulars; and each is just itself, identical with none other, different from all else. We may speak of particular difference as numerical difference; but we must not take the difference to be due to counting, the particulars must be there to be counted. This particularity of plurality of nature is a *necessity* character of the real, it is nothing contingent; and to be empirical and scientific is unreservedly to acknowledge this real natural necessity, and

never, either overtly or covertly, to neglect particular difference. (Just as conduct is ethical when it consistently regards individual values, so science is ethical in its deference to particular differences, these being of value for cognition.)

But now, discernible within this plurality of particulars, there may be, and evidently is, a *plurality of kinds*. In other words, the particulars may be similar in diverse respects. Note that the particulars must be different and plural if they are to be similar to one another. A particular cannot be similar to itself, it can and must be identical with itself. Identity is not close similarity. As the particulate character of nature is its real plurality, so the plurality of diverse kinds is the *specificity* of nature. The plurality of particulars is a necessary character, but the plurality of kinds is a contingent character. The particulars must be different, they may be (and often observably are) similar. The contingent specificity of nature requires the unconditional particularity of nature. Cosmos still, and of necessity, rides upon chaos.

The last two paragraphs present the metaphysics of science, reason, and common sense, in that they comprise the whole of necessary truth. This is all the metaphysics we can have or need to know.

The cognition of a world so differently particulate and diversely specific clearly requires two sorts of knowledge. First, there must be knowledge of particulars as such - an absolute knowledge of the actual and necessary plurality of nature. This knowledge grounds every construction of mathematical symbolism and directs all our use of this in quantitative description. Because it constitutes our purchase upon the real plurality of nature, we may speak of it as *number knowledge*; but we must be careful not to confuse it with our knowledge of number theory, which is a special knowledge of certain symbolic artifacts used in counting. Our knowledge of multiplication tables may help us to know that five corrals each containing eight horses together contain forty horses; but the knowledge so gained is of plural horses, not of multiplication tables or number terms. We see that descriptive science does include an absolute knowledge of the necessary plurality of nature, which is what makes counting possible.

Secondly, there may and should be knowledge of the many similarities presented by the plural particulars, especially of the more widespread and repeated similarities that we call *forms, types, structures, uniformities, laws, constants*, etc. This is the knowledge delivered by the special sciences, properly so-called because each is bent upon certain only of these specific similarities. It is important to remember that all of this special or specific knowledge is a knowledge of the *possible*, in that particulars, necessarily different, may be but need not be similar. While we must require particulars to be different, we can only anticipate their being similar; and by attending

to what is is more often similar, we come to distinguish remote possibilities from those which are more probable. Here, then, are the special knowledges, always of the contingent and possible, gained by comparison or induction. As true scientists, we acknowledge that whatever is specific and possible is contingent upon what is particular and necessary. We think of particulars as being possibly specified. We do not think, as does the philosopher, of specific forms or structures being necessarily realized in particular instances. We know that the actual and particular determines the possible and specific. (Into the important question of how particulars determine natural kinds, we we must not enter here.)

There are many particulars of many kinds! Should we have expected to locate reason in a truth less simple, familiar, or comprehensive? This is rational knowledge, affirmed and applied in every exercise of thought whatsoever; and it shows us (if we honestly wish to learn) why all our special and probable knowledge of what is specific in nature is necessarily grounded upon an absolute knowledge of what is particular and plural in nature. There could not be many kinds if there were not many particulars. There could not be even one kind. Why have false philosophies arisen and persisted, to make confused and recondite what is so simple and clear? What axe does the philosopher grind?

The source of every philosophical misrepresentation of science and nature lies in a perverse but habitual confusion of natural knowledge, which is always of particulars in their differences and likenesses, with a special knowledge of the symbolic artifacts implementing expression. The more elaborate of these symbolic constructions we call *theories*; and the error of the philosopher, be he rationalist or empiricist, is to put "theoretical knowledge" cognizant only of symbolism into the place of the natural knowledge communicated via symbolism. This is a confusion of language with fact. With this common error the rationalist and the empiricist compound their partial but complementary insights, each achieving a distortion of the truth.

The rationalist is properly aware that we possess an absolute and authentic knowledge of the plurality of the real. He knows that when the scientist describes binary stars and five-petalled flowers, or states that earth has but one moon while another planet has a dozen or more, the number terms used in such descriptions are no less descriptive (they are more exactly descriptive) than are the other terms, *star*, *flower*, *planet*, and *moon*. He knows that four cats are as truly four as they are feline; he knows that differences are as real as likenesses. What is so patently descriptive as number knowledge? There is just nothing this side of heaven or hell that can't be counted, measured, quantitatively assayed, or statistically surveyed. Yet what we engage here is something absolute. We can be

quite sure, if we add five and eight to get twelve or fourteen, that we have somewhere overlooked real particular differences; and we know that when a prediction fails of verification, it is some special hypothesis, not universal arithmetic, that requires modification. But now the rationalist confuses this absolute and universal knowledge of real plurality with his knowledge of number theory, this latter being a special and no more than probable knowledge of something specifically symbolic and human. He accordingly believes himself to have absolute knowledge of a specific form or structure that "necessarily" invests universal being - a knowledge quite impossible, inasmuch as the specific is necessarily the possible or probable. And he now attributes what is specifically symbolic - theoretical or logical structure - to nature at large. "In the beginning was language, the Word," he says; "and the Word was made flesh and stone and whatever particularity exists." He holds the material interactions of nature to be realizations of certain possibilities of symbolic expression; he finds everywhere a "universe of discourse"; he allows only symbols to be real; he takes grammar to be the ultimate law of universal nature.

The empiricist finds this idealistic metaphysics to be fantastic. Unlike the rationalist, he is quite aware that whatever is specific can only be probably known, never absolutely known. Yet he, too, confuses number knowledge with number theory, so that he has to explain what gives to this special and probable knowledge of numerical symbolism the appearance of being absolute and necessary. He does this by an ostensible "reduction" of number theory to logical theory. This logistical construction would show numerical necessity to be nothing else than "logical necessity," which can be understood as no absolute necessity but only a condition of clear and intelligible exposition. He concludes that number knowledge and mathematical science form no part of descriptive knowledge. The mathematical terms which stud scientific descriptions refer to nothing in external nature. They are only a part of the vehicle of symbolic description, and their whole reference is to certain volitional decisions prescribing how terms shall be used.

One must wonder whether the logical empiricist, who would have us never say more than we mean, comprehends what he is saying or realizes the enormity of his misinterpretation of science. He wishes to invalidate the claim to absolute knowledge in order to justify belief in an inductive science cognizant of what is specific and probable in nature. Yet how can he believe in the descriptive power of physical science if he denies descriptive cogency to all quantitative distinctions? Is light not really speedier than sound? Is the sun not really more massive than the earth? Can we see in all of the measurable differences of things only an illusory reflection of our mode of symbolic expression? And is the irresistible gravitation of

science to exact, quantitative, and statistical description a consequence not of the nature of nature (everywhere particulate and plural), but only of our felicities of speech? Truly, we must judge that of the two philosophical misrepresentations it is rationalism that stays closer to scientific truth. Its idealistic fantasy is less absurd than this positivistic unbelief in the particulate, plural, necessarily quantitative character of the real.

For the scientist, rationalism and empiricism are but two expressions of the same truth; they are not antithetical doctrines. The empirical emphasis upon particular occurrence as the source and criterion of general knowledge is equally an emphasis upon the actual and necessary plurality of nature, which is what allows and requires an absolute and purely rational number knowledge. A science that is less than exact, and that makes no use of quantitative method and mathematical implementation, remains less than empirical.

If science knows only this numerical necessity that requires reality to be particularly differentiated and plural, the working logic of science has always been and must forever remain mathematics. There can be no necessity imposed upon thought and cognition that is not imposed upon nature at large - we cannot suppose nature to be free and only man confined. Mathematical science, it is true, contains more than a sheer knowledge of the plurality of the real - it includes a knowledge of the symbolism designed to bring that knowledge to expression. This mathematical symbolism is far and away the mightiest artifact created by man, and the most consequential. Its construction has occupied several millennia; and its whole purpose is to give expression in innumerable, specific ways to man's rational knowledge of numerical necessity. The mathematician owes nothing to formal logic; nor may he recognize the claim of the logician to some impossible insight into "formal," "logical," or other "specific necessity"- a necessity that cannot be.

This fiction of "logical necessity" we owe to Aristotle, who believed that the mathematically implemented science of his predecessors did less than justice to the rich specificity of nature. To remedy this neglect, Aristotle proposed to substitute for mathematics, then as now the true logic of science, another discipline which would enable the scientist to disregard particular numerical differences and proceed directly to a knowledge of what is specific in nature. (For man possesses, Aristotle mistakenly believed, an absolute rational knowledge of specific forms.) All that Aristotle's new logic could or did attempt was a puerile substitution for the exact quantifications of mathematical science (, 1, 2, 3, 4, ...) of the rough and inadequate quantifications of prescientific speech (*some, none, all*). Yet this atavistic Aristotelian logic, presumably because it promised an easy short-cut to scientific truth, has come down all of the centures to ourselves, to confound reason and defeat science. In the

Middle Ages it spawned a vast pseudoscience compacted of verbal pedantry; and today it generates a new scholastic pedantry, promulgated by those who still feel themselves compelled by "logical necessity" to derive the exact quantifications of mathematics from the rough and casual quantifications *some, none, all,* and who are still persuaded that description, to be precise, must "of logical necessity" be inexact.

However, modern logic begins to emancipate itself from Aristotelian error and to become what it ought to be, namely a special science included within anthropology but bent upon what is specifically symbolic in human behavior. This science of symbolics will not profess, as did formal logic, to have authoritative knowledge of a "logical necessity" imposed upon thought and cognition; but it will nevertheless help us to distinguish between what is specifically symbolic in scientific exposition, what has reference to specific similarities in the nature described, and what is imposed upon thought and cognition and expression, as upon everything else, by the necessary particularity and plurality of being. As these distinctions are clarified, the true status of mathematics as a descriptive yet absolute science will become apparent and be given the recognition that is its due.

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(OUR CONTRIBUTORS, *continued from back of contents*)

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ON THE INTRINSIC DERIVATIVE OF GENERALIZED ORDER

Hiroyoshi Sasayama

The fact due to Prof. Homer V. Craig that the m-th intrinsic derivative $\delta T^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot}/\delta t^m$ of integral higher order m of a tensor $T^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot}$ can be expressed as a contraction of extensors introduced by him, seems to tell us something of the geometrical importance of extensors. The principal object of this paper is to introduce the intrinsic derivative $\{f(\frac{\delta}{\delta t})\}^t_{t_0} T^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot}$ of the tensor $T^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot}$ as a geometrical generalization of E.L. Post's derivative $\{f(D)\}^t_{t_0}$ such that if $f(D) = D^M$ where M is an arbitrary complex number and particularly a negative integer $-m$ it gives the M-th intrinsic derivative as a generalization of the fractional derivative and the m-th order tensorial integral of tensors respectively. For this purpose, at first, we shall show that the excovariant and crossed extensors of integral infinite ranges are derived from tensors by Post's differentiation and then, from such generalized derived extensors, we construct the required derivatives of covariant tensors in first a nonholonomic space whose original space is affinely connected and treat of their properties.

In the following, the Latin indices a,b,c,d and $h,i,j,k;\ h',i',j',k'$ will indicate non-negative integers with ranges $0, 1, 2, \ldots$ and $1, \ldots n$ respectively and the notation $F^{(a)}$ for any differentiable function F of a single variable indicates the a-th order derivative of F with respect to the variable and the word "P-differentiable" means the differentiability in the sense of E.L. Post's generalized derivative.

1. *Generalized derived extensors of infinite range.*

Let C be a parametrized arc expressed by $x^i = x^i(t)$ with a parameter t in an n-dimensional space X_n referred to the coordinate system (x^i) where every coordinate transformation $\bar{x}^i = \bar{x}^i(x)$ will be supposed to satisfy the condition that the partial derivatives $\partial\bar{x}^i/\partial x^j$'s, $\partial x^j/\partial\bar{x}^i$'s, the weighted Jacobian and their products are analytic functions of t along C with the Taylor series of the radius of convergence greater than $t - t_0$ at t where t_0 is a fixed constant. If the components v^i and v_i of absolute contravariant and

covariant vector fields defined along C are P-differentiable functions of t, then by the generalized Leibniz formula due to E. L. Post, under the coordinate transformation $\bar{x}^i = \bar{x}^i(x)$, the quantities

$$\{f^{(a)}(D)\}_{t_0}^{t} v^i/a! \text{ and } \{f^{(a)}(D)\}_{t_0}^{t} v_i/a! \text{ for } a = 0,1,2,\ldots$$

are transformed to

$$\{f^{(a)}(D)\}_{t_0}^{t} \bar{v}^i/a! = a^{-1} \sum_{b=0}^{\infty} b!^{-1} \frac{\partial \bar{x}^{i(b)}}{\partial x^j} \{f^{(a+b)}(D)\}_{t_0}^{t} v^j$$

$$= \sum_{b'=a}^{\infty} \binom{b'}{a} \frac{\partial \bar{x}^{i(b'-a)}}{\partial x^j} \{f^{(b')}(D)\}_{t_0}^{t} v^j/b'!$$

and

$$\{f^{(a)}(D)\}_{t_0}^{t} \bar{v}_i/a! = \sum_{b=a}^{\infty} \binom{b}{a} \frac{\partial x^{j(b-a)}}{\partial \bar{x}^i} \{f^{(b)}(D)\}_{t_0}^{t} v_j/b!$$

respectively. Therefore, if the qunatities v_a^i and v_{ai} transform as

$$\bar{v}_a^i = \sum_{b=a}^{\infty} \frac{\partial \bar{x}^{(b)i}}{\partial x^{(a)j}} v_b^j \text{ and } \bar{v}_{ai} = \sum_{b=a}^{\infty} \frac{\partial x^{(b)j}}{\partial \bar{x}^{(a)i}} v_{bj},$$

where the infinite series in the right hand sides are supposed to converge, under the coordinate transformation $\bar{x}^i = \bar{x}^i(x)$, they will be called components of crossed and ordinary excovariant exvectors of integral infinite range, thus we have the following:

THEOREM (1.1) *If the components v^i and v_i of absolute contravariant and covariant vector fields along C are P-differentiable in t, then the quantities*

$$v^i_{\{a\}} = \{f^{(a)}(D)\}_{t_0}^{t} v^i/a! \quad \text{and}$$

$$v_{\{a\}i} = \{f^{(a)}(D)\}_{t_0}^{t} v_i/a! \quad (a = 0,1,2,\ldots) \tag{1.1}$$

are components of crossed and ordinary excovariant exvectors of integral infinite range respectively.

Next, for the absolute tensor fields G_{ij}, G^i_j, G^{ij} of rank 2 defined along C which are P-differentiable in t, let us consider the quantities $(a!\ b!)^{-1}\{f^{(a+b)}(D)\}_{t_0}^{t} G_{\cdot\cdot}^{\cdot\cdot}$ for $a,b = 0,1,2,\ldots$ Then, under the coordinate transformation $\bar{x}^i = \bar{x}^i(x)$, these quantities e.g. $(a!\ b!)^{-1}\{f^{(a+b)}(D)\}_{t_0}^{t} G_{ij}$ are transformed to

$$(a!\ b!)^{-1}\{f^{(a+b)}(D)\}_{t_0}^{t} \bar{G}_{ij} =$$

$$= (a!\ b!)^{-1} \sum_{c=0}^{\infty} c!^{-1}\{f^{(a+b+c)}(D)\}_{t_0}^{t} G_{hk} \frac{\partial x^h}{\partial \bar{x}^i} \frac{\partial x^k}{\partial \bar{x}^j}^{(c)}$$

$$= (a!b!)^{-1} \sum_{c=0}^{\infty} \sum_{d=0}^{\infty} c!^{-1}\binom{c}{d}\{f^{(a+b+c)}(D)\}_{t_0}^{t} G_{hk} \frac{\partial x^{h(c-d)}}{\partial \bar{x}^i} \frac{\partial x^{k(d)}}{\partial \bar{x}^j}$$

$$= \sum_{c'=a}^{\infty} \sum_{d'=b}^{\infty} \binom{c'}{a}\binom{d'}{b} \frac{\partial x^{h(c'-a)}}{\partial \bar{x}^i} \frac{\partial x^{k(d'-b)}}{\partial \bar{x}^j} (c'!\,d'!)^{-1}\{f^{(c'+d')}(D)\}_{t_0}^{t} G_{hk}$$

assuming the absolute covergence of these series, i.e.

$$(a!\ b!)^{-1}\{f^{(a+b)}(D)\}_{t_0}^{t} \bar{G}_{ij} =$$

$$= \sum_{c'=a}^{\infty} \sum_{d'=b}^{\infty} \frac{\partial x^{(c')h}}{\partial \bar{x}^{(a)i}} \frac{\partial x^{(d')k}}{\partial \bar{x}^{(b)j}} (c'!\,d'!)^{-1}\{f^{(c'+d')}(D)\}_{t_0}^{t} G_{hk}$$

Similarly, we have

$$(a!\ b!)^{-1}\{f^{(a+b)}(D)\}_{t_0}^{t} \bar{G}^i_j =$$

$$= \sum_{c=a}^{\infty} \sum_{d=b}^{\infty} \frac{\partial \bar{x}^{(c)i}}{\partial x^{(a)h}} \frac{\partial x^{(d)k}}{\partial \bar{x}^{(b)j}} (c!\ d!)^{-1}\{f^{(c+d)}(D)\}_{t_0}^{t} G^h_k$$

and

$$(a!\ b!)^{-1}\{f^{(a+b)}(D)\}_{t_0}^{t} \bar{G}^{ij} =$$

$$= \sum_{c=a}^{\infty} \sum_{d=b}^{\infty} \frac{\partial \bar{x}^{(c)i}}{\partial x^{(a)h}} \frac{\partial \bar{x}^{(d)j}}{\partial x^{(b)k}} (c!\ d!)^{-1}\{f^{(c+d)}(D)\}_{t_0}^{t} G^{hk}$$

In general, we have

THEOREM (1.2) *If the components $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ of an absolute tensor field and a relative scalar S of weight pw defined along C are P-differentiable in t, then the quantities*

$$(1.2) \qquad T^{i_1\cdots i_p}_{\{a_1\}\cdots\{a_p\}\{b_1\}j_1\cdots\{b_q\}j_q} =$$

$$\left(\prod_{r=1}^{p} a_r! \prod_{s=1}^{q} b_s!\right)^{-1} \{f^{(a_1+\cdots+a_p+b_1+\cdots+b_q)}(D)\}_{t_0}^{t}\, T^{i_1\cdots i_p}_{j_1\cdots j_q}$$

and

$$S_{\{a_1+\dots+a_p\}} = \left(\prod_{r=1}^{p} a_r!\right)^{-1} \{f^{(a_1+\dots+a_p)}(D)\}_{t_0}^{t} S \tag{1.3}$$

are components of an extensor and a Jacobian extensor of integral infinite range and the types indicated by their indices, i.e., under the coordinate transformation $\bar{x}^i = \bar{x}^i(x)$, *these are transformed as*

$$\bar{T}^{i_1\dots i_p}_{\{a_1\}\dots\{a_p\}\{b_1\}j_1\dots\{b_q\}j_q} = \tag{1.4}$$

$$\sum_{c_1=a_1}^{\infty} \dots \sum_{c_p=a_p}^{\infty} \sum_{d_1=b_1}^{\infty} \dots \sum_{d_q=b_q}^{\infty} \prod_{r=1}^{p} \cdot$$

$$\frac{\partial \bar{x}^{(c_r)i_r}}{\partial x^{(a_r)h_r}} \prod_{s=1}^{q} \frac{\partial x^{(d_s)k_s}}{\partial \bar{x}^{(b_s)j_s}} T^{h_1\dots h_p}_{\{c_1\}\dots\{c_p\}\{d_1\}k_1\{d_q\}k_q}$$

and

$$\bar{S}_{\{a_1\}\dots\{a_p\}} = \sum_{c_1=a_1}^{\infty} \dots \sum_{c_p=a_p}^{\infty} \prod_{r=1}^{p} \underline{X}^{c_r}_{a_r} S_{\{c_1\}\dots\{c_p\}} \tag{1.5}$$

respectively, where the infinite series involved are assumed to be absolutely convergent and $\underline{X}^{c}_{a} = \binom{c}{a} |\partial x/\partial \bar{x}|^{w(c-a)}$.

2. *The generalized order intrinsic derivative in a non-holonomic space.*

In order to define the intrinsic derivative of generalized order for a covariant tensor, we now consider a non-holonomic space N_n with the base covariant and reciprocal contravariant vectors $\lambda^{i'}_{i}$ and $\lambda^{i}_{i'}$ $(i' = 1,\dots,n)$ respectively whose original n-dimensional point space X_n is an affinely connected space L_n with the symmetric connection parameter $\Gamma^i_{jk}(x)$. For a covariant tensor field $T_{i_1\dots i_p}$ of rank p defined along an arc C in L_n, let $T_{i'_1\dots i'_p} = \prod_{r=1}^{p} \lambda^{i_r}_{i'_r} T_{i_1\dots i_p}$ be the non-holonomic tensor components in N_n and put

$$\lambda^{ai}_{i'} = \lambda^{j}_{i'} L^{ai}_{j} \tag{2.1}$$

where $L^{ai}_j = D_a \delta|^{ai}_j$ be the a-th components of the a-th lower extensive derivative of the Kronecker delta δ^i_j introduced by H.V.Craig, that is,

$$L^{0i}_j = \delta^i_j, \quad L^{b+1i}_j = L^{bi(1)}_j - L^{bi}_k \Gamma^k_{hj} x^{(1)L} \quad (0 \le b \le a).$$

Then, the quantities

(2.2)

$$\{f(\frac{\delta}{\delta t})\}_{t_0}^{t} T_{i'_1\ldots i'_p} = \sum_{a_1=0}^{\infty} \sum_{a_p=0}^{\infty} \prod_{r=1}^{p} \lambda^{a_r i_r}_{i'_r} T_{\{a_1\}i_1\ldots\{a_p\}i_p}$$

where the multiple infinite series in the right hand side be supposed to regularly converge, will be referred to as the *intrinsic derivative of generalized order of the non-holonomic components* $T_{i'_1\ldots i_p}$ *in* N_n *of a covariant tensor field* $T_{i_1\ldots i_p}$ *along* C *in* L_n. When $f(Z) = Z^M$, M being an arbitrary complex number, and $f(Z) = Z^{-\mu}$, μ being a positive real number, it will be referred to as the *M-th order intrinsic derivative and μ-th order tensorial integral of* $T_{i'_1\ldots i_p}$ and denoted by $\delta^M T_{i'_1\ldots i'_p}/\delta t^M$ and $\overset{t}{\underset{t_0}{I^\mu}}\, T_{i'_1\ldots i'_p}(\delta t)^\mu$ respectively. For a covariant vector v_i ($p=1$) and $a = 0, 1, 2, \ldots$, we have

$$\{f^{(a)}(\frac{\delta}{\delta t})\}_{t_0}^{t} v_{i'} = a! \sum_{b=a}^{\infty} \binom{b}{a} \lambda^{b-aj}_{i'} v_{\{b\}j}. \tag{2.3}$$

If N_n coincides with the original space L_n, then in place of the $\lambda^{ai}_{i'}$ we use L^{ai}_{j}, which becomes $\delta^a_0\, \delta^i_j$ in a coordinate system which is geodesic along the curve in question or in a flat space, hence we have the following

THEOREM (2.1) *The intrinsic derivative of generalized order of a P-differentiable covariant tensor field* $T_{i_1\ldots i_p}$ *defined along* C *in* L_n *is given by the following covariant tensor*

(2.4)

$$\{f(\frac{\delta}{\delta t})\}_{t_0}^{t} T_{i_1\ldots i_p} = \sum_{a_1=0}^{\infty}\cdots\sum_{a_p=0}^{\infty} \prod_{r=1}^{p} L^{a_r j_r}_{i_r} T_{\{a_1\}j_1\ldots\{a_p\}j_p}$$

where the multiple infinite series in the right hand side be supposed to regularly converge, which becomes the Post's generalized derivative $\{f(D)\}_{t_0}^{t} T_{i_1\ldots i_p}$ *of* $T_{i_1\ldots i_p}$ *in a geodesic coordinate system or in a flat space, and hence, coincides with the ordinary m-th intrinsic derivative* $\delta^m T_{i_1\ldots i_p}/\delta t^m$ *when* $f(\frac{\delta}{\delta t}) = \frac{\delta^m}{(\delta t)^m}$, *with m a positive integer.*

COROLLARY (2.1) *The intrinsic derivative of generalized order of a P-differentiable covariant vector field* v_i *along* C *is given by the following covariant vector*

$$\{f^{(a)}(\frac{\delta}{\delta t})\}_{t_0}^{t} v_i = a! \sum_{b=a}^{\infty} \binom{b}{a} L^{b-aj}_{i} V_{\{b\}j} \quad (a = 0, 1, 2, \ldots) \tag{2.5}$$

COROLLARY (2.2) *The M-th intrinsic derivative of a covariant tensor field* $T_{i_1\ldots i_p}$ *fractionally differentiable in t along C is given by*

$$(2.6)\qquad \frac{\delta^M T_{i_1\ldots i_p}}{\delta t^M} = \sum_{a_1=0}^{\infty}\ldots\sum_{a_p=0}^{\infty}\prod_{r=1}^{p} L_{i_r}^{a_r j_r}\, T^{(M)}_{(a_1)j\ \ldots(a_p)j_p}$$

where, for a complex number M,

$$(2.7)\qquad T^{(M)}_{(a_1)j_1\ldots(a_p)j_p} = \binom{M}{a_1\ldots a_p} T^{(M-a_1-\ldots-a_p)}_{j_1\ldots j_p}$$

$$(2.8)\qquad \binom{M}{a_1\ldots a_p} = \frac{M(M-1)\ \ldots\ (M+1-a_1-\ldots-a_p)}{a_1!\ a_2!\ \ldots\ a_p!}$$

and the multiple infinite series in the right hand side be supposed to regularly converge.

COROLLARY (2.3) *The* μ*-th order tensorial integral of a covariant tensor field* $T_{i_1\ldots i_p}$ *fractionally integrable along C is given by*

$$(2.9)\qquad \overset{t}{\underset{t_0}{I}}{}^{\mu}\, T_{i_1\ldots i_p}\ (\delta t)^{\mu} = \sum_{a_1=0}^{\infty}\ \ldots$$

$$\sum_{a_p=0}^{\infty}\prod_{r=1}^{p} L_{i_r}^{a_r j_r}\binom{-\mu}{a_1\ldots a_p}\int_{t_0}^{t}{}^{\mu+a_1+\ldots+a_p} T_{j_1\ldots j_p}(dt)^{\mu+a_1+\ldots+a_p}$$

where $\int_{t_0}^{t}{}^{M}\ T^{\cdots}_{\cdots}(dt)^M$ *denotes the M-th order fractional integral of* $T^{\cdots}_{\cdots}$ *and the multiple infinte series in the right hand side be supposed to regularly converge.*

If $\lambda_i^{i'}$ be analytic functions in t along C with the Taylor series of radius of convergence greater than $t - t_0$ at t, is (2.3) using the relations

$$(2.10)\qquad v_{\{b\}i} = \sum_{b'=b}^{\infty} \lambda^{(b')j'}_{(b)i}\, v_{\{b'\}j'}$$

where $\lambda^{(b')j'}_{(b)i} = \binom{b'}{b}\lambda_i^{j'(b'-b)}$, which are easily derived from $v_i = \lambda_i^{j'}\ v_{j'}$, we get

$$\{f^{(a)}(\tfrac{\delta}{\delta t})\}_{t_0}^{t}\ v_{i'} = a!\sum_{b=a}^{\infty}\ \sum_{b'=b}^{\infty}\binom{b}{a}\lambda_{i'}^{b-ai}\ \lambda^{(b')j'}_{(b)i}\ v_{\{b'\}j'}$$

$$= a! \sum_{b'=a}^{\infty} \left[\sum_{b=a}^{b'} \binom{b}{a} \lambda_{i'}^{b-a\,i} \lambda_{(b)i}^{(b')j'} \right] v_{\{b'\}j'}$$

$$= a! \sum_{b'=a}^{\infty} \binom{b'}{a} \left[\sum_{b=a}^{b'} \binom{b'-a}{b-a} \lambda_{i'}^{b-a\,i} \lambda_{i}^{j'(b'-b)} \right] v_{\{b'\}j'},$$

hence we have the following

THEOREM (2.2) *If the base covariant vector $\lambda_i^{i'}$ be analytic functions in t along C with the Taylor series of radius of convergence greater than $t - t_0$ at t, and if the series $\sum_{b=a}^{\infty} \sum_{b'=b}^{\infty} \binom{b}{a} \Lambda_{i'}^{b-a\,i} \Lambda_{(b)i}^{(b')j'} v_{\{b'\}j'}$ be absolutely convergent, then the intrinsic derivative of generalized order of the non-holonomic components $v_{i'}$ in N_n of a covariant vector field v_i P-differentiable along C in L_n can be expressed in terms of the non-holonomic components in N_n as follows:*

$$\{f^{(a)}(\tfrac{\delta}{\delta t})\}_{t_0}^{t} v_{i'} = a! \sum_{b'=a}^{\infty} \binom{b'}{a} N_{i'}^{b'-a\,j'} v_{\{b'\}j'} \tag{2.11}$$

where

$$N_{i'}^{b'j'} = \sum_{c=0}^{b'} \lambda_{i'}^{c\,i} \lambda_{(c)i}^{(b')j'} \tag{2.12}$$

which are called the intrinsic derivation coefficients of the non-holonomic space N_n.

Finally, the intrinsic derivative of generalized order of the metric covariant tensor g_{ij}, in a Riemannian space R_n coincides with $g_{ij}\{f(D)\}_{t_0}^{t}1$, in a Fermi geodesic coordinate system, hence in every coordinate system. Thus, we have the following k

THEOREM (2.3) *In R_n, the intrinsic derivative $\{f(\frac{\delta}{\delta t})\}_{t_0}^{t} g_{ij}$ of generalized order of the covariant metric tensor g_{ij} is equal to $g_{ij}\{f(D)\}_{t_0}^{t}1$.*

COROLLARY (2.3) *(Generalized Ricci's lemma) The intrinsic derivative $\delta^M g_{ij}/\delta t^M$ of arbitrary order M of g_{ij} in R_n is equal to $g_{ij}\, 1^{(M)}$, which vanishes for a positive integer $M = m > 0$.*

COROLLARY (2.4) *The intrinsic derivative $(\delta^\mu g_{ij})/(\delta t^\mu)$ of non-integral real order μ of g_{ij} in R_n does not vanish unless $\mu > 0$ and $t = t_0$.*

3. The generalized Post's derivation operator.

We shall investigate the properties of our derivation of generalized order introduced above. If we define the intrinsic derivative of generalized order of an absolute scalar S as its Post's derivative, i.e.,

$$\{f(\frac{\delta}{\delta t})\}_{t_0}^{t} S = \{f(D)\}_{t_0}^{t} S, \tag{3.1}$$

then we can give properties of our derivation as an operator by the following theorems; all of which follow immediately from the corresponding theorems in E. L. Post's differentiation, by taking a a coordinate system which is geodesic along the curve in question.

THEOREM (4.1) *The intrinsic derivation* $\{f(\frac{\delta}{\delta t})\}_{t_0}^{t}$ *of generalized order is a linear operator i.e.,*

$$\{f(\frac{\delta}{\delta t})\}_{t_0}^{t} (\sum_r c_{(r)}{}^{(r)}T\dots) = \sum_r c_{(r)}\{f(\frac{\delta}{\delta t})\}_{t_0}^{t} {}^{(r)}T\dots \tag{4.2}$$

where $C_{(r)}$ *be constants and* ${}^{(r)}T^{\dots}_{\dots}$ *P-differentiable scalars or vectors or covariant tensors of the same type and weight* 0.

THEOREM (4.2) *(Generalized Leibniz formula).*

$$\{f(\frac{\delta}{\delta t})\}_{t_0}^{t} (T\dots U^{\dots}_{\dots}) = \sum_{a=0}^{\infty} (a!)^{-1}\{f^{(a)}(\frac{\delta}{\delta t})\}_{t_0}^{t} T\dots \frac{\delta^a U\dots}{\delta t^a},$$

where $T^{\dots}_{\dots}$ *(or* $U^{\dots}_{\dots}$ *) be an absolute contravariant vector and* $U^{\dots}_{\dots}$ *(or* $T^{\dots}_{\dots}$ *) and absolute scalar; or both* $T^{\dots}_{\dots}$ *and* $U^{\dots}_{\dots}$ *be covariant tensors or scalars of weight* 0, *provided that* $\{f(D)\}_{t_0}^{t} T^{\dots}_{\dots}$ *exist and* $U^{\dots}_{\dots}$ *are analytic functions of* t *with the Taylor series of the radius of convergence greater than* $t - t_0$ *at* t.

THEOREM (4.3) *For an absolute vector or covariant tensor field* $T^{\dots}_{\dots}$ *along* C *the following law of the successive intrinsic derivation of generalized order holds good:*

$$\{f_1(\frac{\delta}{\delta t})\}_{t_0}^{t} \{f_2(\frac{\delta}{\delta t})\}_{t_0}^{t} T\dots = \{f_1(\frac{\delta}{\delta t})\, f_2(\frac{\delta}{\delta t})\}_{t_0}^{t} T\dots \tag{4.5}$$

provided that

$$\{f_1(D)\}_{t_0}^{t} \{f_2(D)\}_{t_0}^{t} T\dots = \{f_1(D)\, f_2(D)\}_{t_0}^{t} T\dots \tag{4.6}$$

COROLLARY (4.2) *For positive real numbers μ_1, $\mu_2 > 0$, the following formulas consist:*

$$\frac{\delta^{\mu_2}}{\delta t^{\mu_2}} \left[\overset{t}{\underset{t_0}{I}}{}^{\mu_1} T^{\cdots}_{\cdots} (\delta t)^{\mu_1} \right] = \frac{\delta^{\mu_2-\mu_1} T^{\cdots}_{\cdots}}{\delta t^{\mu_2-\mu_1}} \tag{4.7}$$

$$\overset{t}{\underset{t_0}{I}}{}^{\mu_2} \left[\overset{t}{\underset{t_0}{I}}{}^{\mu_1} T^{\cdots}_{\cdots} (\delta t)^{\mu_1} \right] (\delta t)^{\mu_2} = \overset{t}{\underset{t_0}{I}}{}^{\mu_1+\mu_2} T^{\cdots}_{\cdots} (\delta t)^{\mu_1+\mu_2} \tag{4.8}$$

$$\frac{\delta^{\mu_2}}{\delta t^{\mu_2}} \frac{\delta^{\mu_1} T^{\cdots}_{\cdots}}{\delta t^{\mu_1}} = \frac{\delta^{\mu_1+\mu_2} T^{\cdots}_{\cdots}}{\delta t^{\mu_1+\mu_2}} \tag{4.9}$$

if $\mu_2 = [\mu_2]$ or $0 < \mu_1 < 1$ or $T^{\cdots\,(a)}_{\cdots}(t_0) = 0 \quad (a = 0, 1, \ldots, [\mu_1])$

$$\overset{t}{\underset{t_0}{I}}{}^{\mu_2} \frac{\delta^{\mu_1} T^{\cdots}_{\cdots}}{\delta t^{\mu_1}} (\delta t)^{\mu_2} = \frac{\delta^{\mu_1-\mu_2} T^{\cdots}_{\cdots}}{\delta t^{\mu_1-\mu_2}} \tag{4.10}$$

if $0 < \mu_1 < 1$ or $T^{\cdots\,(a)}_{\cdots}(t_0) = 0 \quad (a = 0, 1, \ldots [\mu_1])$

where [ ] *be the Gaussian symbol denoting the integral part.*

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HOW TALL ARE YOU?

The records we have made on our driver's licenses and other identification papers are indeterminate because ones height depends on the time of day the measurement is made. In mathematical terms, it is dependent on two variables i.e. it is two dimensional.

Males are $\frac{3}{4}''$ taller and females $\frac{1}{2}''$ taller in the morning than in the evening.

It would be interesting to measure a large number of people many times a day and plot the average daily-shrinking and nightly-expanding curves. It would be a sensible guess that these curves would depend secondarily on age, race, diet, employment and infinitely many other variables as well as primarily upon the elasticity of the spinal column*.

Editor

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* See *De Ruby Acta Orthop Scandinavica*, 6:338 - 347, 1935.

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PSEUDO-MULTIPLICATIVE FUNCTIONS

Richard R. Goldberg

This paper is concerned with a class of non-decreasing, non-negative functions $f(x)$ such that

$$f(xy) \geq f(x)\ f(y).$$

Theorems of various types are proved including some on the rate of growth of such functions. All methods used are elementary.

Def. 1 Let E be a set of non-negative real numbers which is closed under multiplication (i.e. $x,\ y \in E$ implies $xy \in E$). The class $P(E)$ of pseudo-multiplicative functions on E is defined as the class of all $f(x)$ such that

*P*1	$f(x) \geq 0$	$x \in E$
*P*2	$f(x)$ is non-decreasing on E	
*P*3	$f(xy) \geq f(x)\ f(y)$	$x,\ y \in E$

If $f(x)$ satisfies *P*1, *P*2, *P*3 we write $f(x) \in P(E)$.

One class of examples is $f(x) = x^r$ for any non-negative r, $E = (0 \leq x < \infty)$.

Also, since

$$xy \geq x + y \qquad (x,\ y \geq 2)$$

we have

$$e^{xy} \geq e^{x+y} = e^x e^y \qquad (x,\ y \geq 2)$$

and thus

$$e^x \in P(2 \leq x < \infty).$$

The next theorem shows that the product of functions of class $P(E)$ is again of class $P(E)$.

TH. 2. If

$$f_1(x),\ f_2(x) \in P(E)$$

then

$$f\ (x),\ f\ (x) \in P(E).$$

Proof: Let $g(x) = f_1(x)\ f_2(x)$. Then $g(x)$ certainly satisfies *P*1 and *P*2. Also, for any $x,\ y \in E$,

$$g(xy) = f_1(xy)\ f_2(xy) \geq f_1(x)\ f_1(y)\ f_2(x)\ f_2(y)$$
$$= f_1(x)\ f_2(x)\ f_1(y)\ f_2(y) = g(x)\ g(y).$$

Thus $g(x)$ satisfies $P3$.

The proofs of the next two theorems are equally easy and are omitted.

TH. 3. If

1. $f(x) \in P(E)$
2. $r \geq 0$.

then
$$[f(x)]^r \in P(E).$$

TH. 4. If

1. $f_n(x) \in P(E) \qquad n = 1, 2, \ldots$
2. $\lim\limits_{n\to\infty} f_n(x) = f(x)$

then
$$f(x) \in P(E).$$

The composition of two pseudo-multiplicative functions is pseudo-multiplicative.

TH. 5.

1. $f_1(x) \quad P(E)$
2. $f_2(x) \quad P(F)$ where F contains the range of $f_1(x)$ and is closed under multiplication.

then
$$f_2[f_1(x)] \in P(E)$$

Proof: Let $g(x) = f_2[f_1(x)]$. Then $g(x)$ satisfies $P1$ and $P2$. Since
$$f_1(xy) \geq f_1(x)\, f_1(y) \quad . \quad (x, y \in E)$$
we have
$$\begin{aligned} g(xy) &= f_2[f_1(xy)] \geq f_2[f_1(x) f_1(y)] \\ &\geq f_2[f_1(x)] \cdot f_2[f_1(y)] = g(x) g(y), \end{aligned}$$
and hence $g(x)$ satisfies $P3$.

TH. 6. If
$$f_1(x),\ f_2(x) \in P(E)$$
then
$$\min\,[f_1(x),\ f_2(x)] \in P(E).$$

Proof: Let $g(x) = \min\,[f_1(x),\ f_2(x)]$. Then $g(x)$ satisfies $P1$ and $P2$. Also
$$g(xy) = \min\,[f_1(xy),\ f_2(xy)] \geq \min[\,f_1(x)f_1(y), f_2(x)f_2(y)]$$
$$\geq \min\,[f_1(x),\ f_2(x)] \cdot \min\,[f_1(y),\ f_2(y)] = g(x)g(y),$$

so that $g(x)$ satisfies $P3$.

We now prove that given any increasing function there exists a pseudo-multiplicative function that increases much faster. Specifically,

TH. 7. Let $h(x)$ be any function non-negative and non-decreasing on $(1 \le x < \infty)$. Then there exists a function $f(x) \in P(1 \le x < \infty)$ such that

$$h(x) = o[f(x)] \qquad (x \to \infty)$$

Proof: Let

$$f(x) = x^{h(x)} \qquad (1 \le x < \infty).$$

Certainly $f(x)$ satisfies $P1$ and $P2$. Also, for $x,\ y \ge 1$,

$$f(xy) = (xy)^{h(xy)} \ge x^{h(x)}\ y^{h(y)} = f(x)\ f(y).$$

Hence $f(x) \in P(1 \le x < \infty)$. Moreover

$$\frac{h(x)}{f(x)} = \frac{e^{\log h(x)}}{e^{h(x)\log x}} \to 0 \qquad (x \to \infty)$$

The next two theorem are of a deepter nature.

TH. 8. Let $f_1(x),\ f_2(x),\ \ldots$ be a sequence of functions of class $P(1 \le x < \infty)$ such that

$$(1)\quad f_n(x) \le f_{n+1}(x) \qquad (1 \le x < \infty;\ n = 1, 2, \ldots)$$

$$(2)\quad f_n(x) = o[f_{n+1}(x)] \qquad (x \to \infty;\ n = 1, 2, \ldots)$$

Then there exists a function $F(x) \in P(1 \le x < \infty)$ such that

$$f_n(x) = o[F(x)] \qquad (x \to \infty;\ n = 1, 2, \ldots)$$

(Roughly, given a sequence of functions in $P(1 \le x < \infty)$ such that each increases faster than its predecessor then we can find a function in $P(1 \le x < \infty)$ which increases faster than *all* members of the sequence.)

Proof: Let

$$F(x) = f_{[x]}(x) \qquad (1 \le x < \infty)$$

where $[x]$ is the greatest integer not exceeding x. Then if $x_1 < x_2$,

$$F(x_1) = f_{[x_1]}(x_1) \le f_{[x_2]}(x_1) \le f_{[x_2]}(x_2) = F(x_2)$$

so that $F(x)$ is non-decreasing. Certainly $F(x) \ge 0$. If $x,\ y \ge 1$ then

$$F(xy) = f_{[xy]}(xy) \geq f_{[xy]}(x)\, f_{[xy]}(y)$$

and thus, by property (1),

$$F(xy) \geq f_{[x]}(x)\, f_{[y]}(y) = F(x)\, F(y)$$

Hence $F(x) \in P(1 \leq x < \infty)$.

Now fix $n \geq 1$. Then for $x > n + 1$

$$f_{n+1}(x) \leq f_{[x]}(x) = F(x)$$

so that

$$\frac{f_n(x)}{F(x)} \leq \frac{f_n(x)}{f_{n+1}(x)} .$$

Thus, by property (2),

$$\frac{f_n(x)}{F(x)} = o\,(1) \qquad (x \to \infty)$$

and the proof is complete.

We conclude by showing that any function of class $P(1 \leq x < \infty)$ that increases to infinity has a minimum rate of increase. Such a function must increase faster than $x^{1/N}$ for some $N > 0$.

TH. 9. If

1. $f(x) \in P(1 \leq x < \infty)$
2. $f(\infty) = \infty$

then there exists an integer $N > 0$ such that

$$x^{1/N} = o[f(x)] \qquad (x \to \infty).$$

Proof: Choose $c > 1$ such that $f(c) \geq 2$. Then by P3, $f(c^n) \geq 2^n$.

Choose a positive integer M such that $c^{2/M} < 2$.

Given $x > c$, let n be the (unique positive) integer such that $c^n \leq x < c^{n+1}$. Then

$$f(x) \geq f(c^n) \geq 2^n \geq c^{2n/M} \geq c^{(n+1)/M} > x^{1/M}$$

so that

$$x^{1/M} < f(x) \qquad (c < x < \infty)$$

and thus for any $N > M$, $x^{1/N} = o[f(x)]$ $(x \to \infty)$.

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THE COMPUTER'S CHALLENGE TO EDUCATION*

Clarence B. Hilberry

When a hundred years from now, someone sits down to write the history of man's technological development, the automatic computer may not rate as the most important of our achievements. I would guess that the discovery of the wheel, for example, is likely to hold its position of intrinsic importance for some time to come. But I think our historian is likely to trace to the computer a new industrial revolution of as great scope as the first Industrial Revolution and coming with vastly increased suddenness. Only 26 years ago, Dr. Bush built the first analog machine, and the first digital computer was put into operation only some twelve years ago. The adaptation of the computer to carry out the information processing functions of business concerns has come only in the last two or three years. Who is to predict what the computer, acting as a giant lever, will do to transform business and industrial processes?

In addition, the computer differs from earlier technological advances in important ways. Earlier tools mechanized a manual process, replacing human fingers and human energy in routine manual tasks. The computer steps to a new level, to mechanize routine *mental* processes. And in addition, it permits us to study the behavior of entire complex systems. And since the information is presented to us with fantastic speed, our actual knowledge of complex systems has greatly increased. Equally important, our capacity for control and prediction and our capacity for insight into these complex systems has also been greatly extended.

The powerful influence of the computer on our common life, therefore, lies in its contribution, in the broadest sense, to science and technology.

Quantitative and logical investigations in science can be carried out on a scale undreamed of only two or three decades ago. Scientific principles and models can be verified against experimental facts with small cost and hazard. Thus, new bodies of scientific knowledge have been established and new bodies of knowledge will be added as a result of research performed on computers.

The amount of engineering research that has been done on analog and digital computers is generally well known. These include development of atomic energy, new power plants for aircraft and automobiles, improved aircraft design, new materials and improvement of other materials to perform under extreme conditions of heat, stress and vibration.

Again in the field of automatic control, the computer has extended our knowledge of self-regulating mechanisms by enabling us to experiment with the principles of feedback in great generality. As a con-

sequence, we have today such fabulous results as the guided missile, the pilotless aircraft, the automatic control devices which run large segments of continuous process industries and intricate manufacturing operations. These studies have also shed new light on the behavior of living organisms, and on information and how it is communicated.

During the past year or two, the computer has come into its own as a record keeper, a data processor and an analyzer of business conditions and trends. Here its potential is vast and its implications far-reaching.

Thus, when up-to-the minute information of a complex operation is fed into a mathematical model by a computer and analyzed there, reliable facts emerge which assist management in making timely and wise decisions. In many cases, these aids in management decision-making are new tools never before available to the manager.

If time permitted, we could illustrate both the actual and the potential contributions of the computer to research in the social science, for example, to our efforts to understand our economy and to find means to prevent ruinous fluctuations in it. Needless to say, great areas of social science research have not to date been explored because, like the physical sciences and technology, they involve problems of extreme complexity far beyond the unaided capacity of any individual or research team to resolve. We may look forward to achievements in the social sciences, as a result of the use of the computer, no less dramatic than those in other fields.

Inherent in the Computation Laboratory, and the programs of instruction and research which spring from it, are not only some of the greatest opportunities which lie before us, but also a series of the most difficult problems which will face higher education in America in the next generation.

Beginning with the Computation Laboratory then, let me mention some of these most pressing problems and perhaps suggest directions in which we might seek the answers to some of them. I hope in the near future that I shall be able to explore some of these ideas in much greater detail with representatives of the business and industrial community.

All the problems are closely interrelated. I mention them here in an order which I hope will have a logical sequence.

The first problem then, is that the totality of knowledge increases these days with enormous rapidity. We have already seen that the computer itself has contributed in no small way to this end. The University faculties find themselves constantly with the necessity of including, by some process, these masses of new information within the educational structure.

The temptation is enormous simply to require that more courses be taken and more time be spent in preparing for any of the college degrees. For practical purposes, the Bachelor's Degree in Engineering now re-

quires four and a half years, and the Colleges of Pharmacy are seriously proposing five years for a Bachelor's Degree in Pharmacy. The extension of the number of months of study is obviously, however, no general answer to this rapid accumulation of new knowledge.

One of the answers lies in the creation of new kinds of high specialization such as the programs springing out of the Computation Laboratory. After a strong undergraduate program, combining general education with a major in mathematics or mathematics and physics, a student can prepare himself, through a Master's Degree, for initial quite heavy responsibilities as a specialist in one aspect or another of this broad field of electronic computation.

In this very fact, however, lies the second of the problems which I'd like to mention to you, for the number of new fields requiring such high specialization is increasing rapidly, and I see no reason to think that the process may not accelerate still further. Only a handful of years ago, no university in the country had a Computation Laboratory, to say nothing of the sequence of educational programs which are now organized around such laboratories.

In each of these new fields of high specialization, demand almost inevitably develops both for work leading to degrees and for a wide variety of in-service education aimed at upgrading present employees in related fields in business and industry. During the years of the life of our Computation Laboratory, for example, 750 students have enrolled for credit courses, and twice that many have studied without credit.

And in addition to the classroom theoretical education, a large number of man and women have acquired mastery of such practical phases of computers as design of components, operation, maintenance, trouble-shooting, mathematical and numerical analysis of a great variety of problems, the development of programming techniques, the analysis of real accounting system, and numerous other actual, technical and business experiences.

Third, it is hardly necessary for me to remark that these programs of high specialization tend to be the most costly of all our educational programs. In the field in which we are most interested, it is obvious that no educational program is possible at all unless we provide and maintain a computer or computers and related equipment.

The fourth problem, like the third, is closely related to the problem of costs. Organic to any university program must be coordinated program of basic research, for research is fundamental to sound instruction, as well as to the outward thrusts which we must always make into the unknown and the unexplored. In the Computation Laboratory, we are proud of the research which has been carried forward in component design, numerical and programming analysis, in mathematical and data processing methods in business. We must find ways to increase the amount of fundamental research which is done across the

University fields.

The fifth problem is of a quite different sort. I can think of no program anywhere in our higher institutions out of which we are turning enough graduates to meet our immediate needs. In many fields, such as education and engineering, we are falling very far behind indeed. Since this lag becomes greater as the momentum of our population increase in this country accelerates, is it not clear that the one problem we don't need to worry about, even though there is a good deal of worry here and there about it, is the problem of our having too many students being educated in our colleges and universities? As far as I can look into the future, I see no reasonable hope of American universities providing adequate numbers of graduates to meet *pressing* social needs.

As a result of this failure of ours to produce enough graduates, all universities are in competition with business and industry and the professions for the men and women of genuinely creative minds, which every university must have on its faculties if it is to produce creative minds in its graduates. This is a problem which concerns business and industry and the professions as much as it concerns the universities. I believe it is true that a genuinely creative intelligence in a university classroom tends to attract and to stimulate creative minds. On the other hand, I believe it is equally true that a mediocre mind tends to produce mediocrity even in minds of higher potential because it places no demands upon them. If a situation develops in which universities cannot retain their full share of the really fine creative minds of every college generation, the effects will, I believe, be all too quickly felt in business and industry itself.

Since it seems to me axiomatic that those in business and industry and we in education are annually going to be less well staffed to do the job which is required of us, how can we find new ways of utilizing the human resources of our organizations with far greater efficiency and effectiveness? I am inclined to think that we can learn in this regard a good deal from the medical profession. The medical profession has pioneered in the creation of a whole series of medical aides and medical technicians. These aides and technicians, adequately trained to carry their responsibilities, relieve the physician of an enormous amount of fairly routine work at the same time that they provide satisfying and rewarding jobs for men and women who might otherwise be doing manual labor.

In visiting one of the computer installations in Detroit recently, I was conducted through the installation and given a running commentary on the operation of the computer by a young man whom, it later turned out, had nover finished high school. It was clear that he had very important responsibilities indeed in relation to the operation of the computer, but they were obviously responsibilities which did not re-

quire the kind of specialized training which a university program provides. Is this not exactly what should happen? Is it not possible that you in business and industry have physicists, chemists, and engineers carrying responsibilities far below those which they should be prepared to carry if university education in these fields is adequate? Do we not perhaps need to sit down together with officers of high schools, and representatives of labor as well, to review our needs, to determine the responsibilities of each agency concerned? For a part of this enormous job of technical training will obviously continue to be done by business and industry itself.

It is clear that the simple competition among us for people is not going to increase the number of people available to us. Is it not possible that out of a new kind of job analysis in which we assume a core of new technical aides carrying most of the routine work and releasing the college educated specialists for creative activities including management responsibilities, – is it not possible that we might hope to find one positive answer to the man power problem which faces us all?

I am by no means sure this would begin to be enough and we need to discuss the whole problem together for it is *one* problem, I believe, not two problems, one of industry and one of education.

Finally, since it is increasingly difficult, as I view the problem at least, to separate the problems of business and industry from those of education, I am moved to ask how can we best promote the sustained cooperative consideration of these mutual problems without which I fear they will not be resolved.

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* A condensation of an address delivered at the opening of the Computation Laboratory's summer program, 1956.

Wayne State University.

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.

DIMENSIONAL ANALYSIS AND HOMOGENEOUS FUNCTIONS

Albert Wilansky

Two straight line cannot enclose a plane region; at least, not a region of finite area. A tentative proof of this is as follows: given two such lines; if they are not parallel, place the origin at their intersection and make a change of scale, bringing each point closer to the origin. This leaves the figure unchanged but decreases the area of the enclosed region - a contradiction. A similar argument deals with parallel lines, the change of scale now being made in the direction of the lines.

While the above remarks leave something to be desired, they cast light on certain aspects of dimensional analysis, as we now show. See (b), below.

(a). Consider the problem: Find the area of the region enclosed by $y^2 = x^3$ and $x = 1$, answer 4/5. One obtains a valuable dimensional check *at every step* (more valuable in more complicated problems) by solving instead the problem: Find the area of the region enclosed by $ay^2 = x^3$ and $x = a$, answer $(4/5)a^2$. Setting $a = 1$ gives the answer to the original problem. There is an abvious sense (defined below) in which the second problem is "dimensionally correct" in its wording, while the first is not. A student who moans at the presence of the letter a in the second problem and says "Why can't the problem be specific?" is berating a useful guide.

By a dimensionally correct equation we mean one of the form $f(a,x,y) = g(a,x,y)$ where f, g are both homogeneous and of the same degree. Any equation can be made dimensionally correct by introduction of a letter e.g. $y = x^2 + \sin x$ becomes $ay = x^2 + a^2 \sin x/a$. A dimensionally correct problem is one in which only dimensionally correct equations occur.

The advantage of dimensional correctness is that one can easily check it at each stage of a computation; for instance, if the quantity $xy^2 + ax$ occured, a mistake would have been made.

(b) Now certain curves appear with a dimensionally correct equation without introduction of a letter a. For example $y = x$, $y = 2x$, $x = 0$, $y = 0$, If the problem mentioned in (a) had been: Find the area of the region enclosed by $y^2 = x^3$ and $y = x$, the only change required to make the problem dimensionally correct would be from y^2 to ay^2. What then would be the situation if the entire boundary of a bounded region consisted of "dimensionally correct" curves, i.e. curves whose equations are dimensionally correct without introduction of extra letters? This is impossible, for a change of scale does not affect the equation of a dimensionally correct curve, hence does not affect the curve, hence does not affect the area of a region enclosed by such curves – a contradiction.

This leads us to the conclusion that *the only dimensionally correct curves are straight lines through the origin,* in particular no non-straight curve is permissible, since such a curve could be made to enclose an area jointly with some line through the origin after appropriate rigid motions. One might seek a counterexample to this result by considering, for example, $xy^4 + y^5 = x^5$, which is dimensionally correct. This is however, merely a set of straight lines through the origin, as one sees immediately by transforming to polar coordinates.

Lehigh University.

VOLUME AND SURFACE OF A SPHERE IN AN N-DIMENSIONAL EUCLIDEAN SPACE

Henry Zatzkis

We shall denote the volume of a sphere by V_N, its surface by S_n and its radius by r. Applying Gauss' theorem to the function ρ^2, where

$$\rho^2 = x_1^2 + x_2^2 + \dots + x_N^2$$

(the x_i are the Cartesian coordinates of the space), we have

$$(1) \qquad \underbrace{\int \dots \int}_{S_N} \nabla^2 \rho^2 \, dV_N = \underbrace{\int \dots \int}_{S_N} \nabla(\rho^2) \cdot d\vec{S}_N \qquad (1)$$

The positive normal on the surface of the sphere is assumed to be directed outward.

Since
$$\nabla^2 f(\rho) = \frac{d^2 f}{d\rho^2} + \frac{(N-1)}{\rho} \frac{df}{d\rho}, \qquad (2)$$

and

$$\nabla f(\rho) = \frac{df}{d\rho}\frac{\vec{\rho}}{\rho} \tag{3}$$

we obtain

$$\underbrace{\int \dots \int}_{V_N} 2N\,dV_N = \underbrace{\int \dots \int}_{S_N} 2\rho\,dS_N \tag{4}$$

or

$$N V_N = r S_N \tag{5}$$

From dimensional reasons we can conclude that

$$V_N = a_N r^N \tag{6}$$

and

$$S_N = b_N r^{N-1} \tag{7}$$

From equation (5) follows

$$b_N = N a_N \tag{8}$$

We have, therefore, to determine the constants a_N. We imagine the sphere V_N intersected by a plane at a distance z from the origin (i.e. $-r \le z \le r$). The intersection is a sphere V_{N-1} of radius $(r^2 - z^2)^{1/2}$. Introducing the polar angle θ $(0 \le \theta \le \pi)$, defined by the equation

$$\cos\theta = \frac{z}{r}, \tag{9}$$

it follows that the volume of the sphere V_{N-1} is given by

$$V_{N-1} = a_{N-1}(r\sin\theta)^{N-1} \tag{10}$$

The sphere $V(r)$ can be built up by "disks" of crossection $V_{N-1}(r\sin\theta)$ and thickness dz.

Hence

$$V_N = a_N r^N = \int_{z=-r}^{z=r} V_{N-1}(r\sin\theta)\,dz$$

$$-\int_{\theta=\pi}^{\theta=0} a_{N-1} r^{N-1}\cdot(\sin\theta)^{N-1} r\sin\theta\,d\theta = 2a_{N-1}r^N \int_{\theta=0}^{\theta=\pi/2} (\sin\theta)^N\,d\theta$$

$$= 2\,a_{N-1}r^N\cdot \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{N+2}{2}\right)} \tag{11}$$

We thus obtain the recursion formulas

$$a_N = \sqrt{\pi}\,\frac{\Gamma\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{N+2}{2}\right)}\,a_{N-1}$$

$$\cdots\cdots\cdots\cdots\cdots \qquad (12)$$

$$a_3 = \sqrt{\pi}\,\frac{\Gamma(2)}{\Gamma(5/2)}\,a_2\,.$$

Multiplying all the terms on the left hand side of equation (12) and equating them to the product of all the terms on the right hand side of the same equation the following relation (after some cancellations) is obtained:

$$a_N = \frac{(\sqrt{\pi})^{N-2}}{\Gamma\left(\frac{N+2}{2}\right)}\,a_2 \qquad (13)$$

But $a_2 = \pi$, and therefore

$$a_N = \frac{(\sqrt{\pi})^N}{\Gamma\left(\frac{N+2}{2}\right)}\,, \qquad (14)$$

and

$$b_N = \frac{N(\sqrt{\pi})^N}{\Gamma\left(\frac{N+2}{2}\right)} \qquad (N = 2, 3, 4, \ldots)$$

Thus the desired formulas are:

$$V_N = \frac{(\sqrt{\pi})^N}{\Gamma\left(\frac{N+2}{2}\right)}\,r^N\,, \qquad (15)$$

and

$$S_N = \frac{N(\sqrt{\pi})^N}{\Gamma\left(\frac{N+2}{2}\right)}\,r^{N-1} \qquad (16)$$

From these formulas the curious result follows that the volume and surface of a sphere of fixed radius r approach zero as the number of dimensions approaches infinity.

Newark College of Engineering

A DIRECT DERIVATION OF THE EQUATION OF THE DIRECTOR CIRCLE OF AN ELLIPSE

A. K. Rajagopal

In an old dairy of the late Prof. A. A. Krisnaswami Ayyangar I found the following proof. I copy it here.

Let $x^2/a^2 + y^2/b^2 = 1$ be the equation of the ellipse. (1)

Let $(x_1 y_1)$, $(x_2 y_2)$ be any two points on it. The tangents at these points are

$$\left.\begin{aligned} xx_1/a^2 + yy_1/b^2 &= 1 \\ xx_2/a^2 + yy_2/b^2 &= 1. \end{aligned}\right\} \tag{2}$$

These are at right angles if

$$x_1x_2/a^4 + y_1y_2/b^4 = 0. \tag{3}$$

The point of intersection of these tangents is given by

$$x = \frac{a^2(y_2 - y_1)}{(x_1y_2 - x_2y_1)}\,; \qquad y = \frac{b^2(x_1 - x_2)}{(x_1y_2 - x_2y_1)} \tag{4}$$

Then

$$x^2 + y^2 = \frac{a^4(y_2 - y_1)^2 + b^4(x_1 - x_2)^2}{(x_1y_2 - x_2y_1)^2} \tag{5}$$

*Since the points lie on the ellipse,

$$b^2x_1^2 + a^2y_1^2 = a^2b^2 \tag{6}$$

$$b^2x_2^2 + a^2y_2^2 = a^2b^2 \tag{7}$$

multiply (6) by y_2^2 and (7) by y_1^2, adding and completing the squares $(x_1y_2 - x_2y_1)$ and $(y_2 - y_1)$

$$b^2(x_1y_2 - x_2y_1)^2 + 2a^2y_1^2y_2^2 = a^2b^2(y_1 - y_2)^2 + 2a^2b^2y_1y_2 - 2b^2x_1x_2y_1y_2$$

or,

$$(x_1y_2 - x_2y_1)^2 - a^2(y_1 - y_2)^2 = 2a^2y_1y_2(1 - x_1x_2/a^2 - y_1y_2/b^2). \tag{8}$$

Similarly,

$$(x_1y_2 - x_2y_1)^2 - b^2(x_1 - x_2)^2 = 2b^2x_1x_2(1 - x_1x_2/a^2 - y_1y_2/b^2). \tag{9}$$

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*From here, it is my proof.

Multiplying (8) by a^2, (9) by b^2 and adding

$$(a^2 + b^2)(x_1 y_2 - x_2 y_1)^2 - a^4(y_1 - y_2)^2 - b^4(x_1 - x_2)^2 =$$

$$= 2(1 - x_1 x_2/a^2 - y_1 y_2/b^2)(a^4 y_1 y_2 + b^4 x_1 x_2) = 0 \qquad \text{by (3).}$$

Therefore from (5) the required locus is

$$x^2 + y^2 = a^2 + b^2.$$

Bengaum, India.

A GENERALIZATION OF WILSON'S THEOREM

Fred G. Elston

When we want to know whether any given whole number p is prime we may apply Wilson's theorem. For according to this theorem p is prime if and only if p is a factor of the expression $(p - 1)! + 1$. Let p be prime, then the following congruence is true:

(1) $$(p - 1)! + 1 \equiv 0 (\mathrm{mod}\ p).$$

We may derive other congruences from (1). We may write (1):

$$(p - 2)!\,(p - 1) + 1 \equiv 0 (\mathrm{mod}\ p).$$

$$(p - 2)!\,(p - 1) + 1 = p(p - 2)! - [(p - 2)! - 1]$$

Since in the right side expression $p(p - 2)!$ is divisible by p, the other term in [ ] is divisible by p. So

(2) $$(p - 2)! - 1 \equiv 0 (\mathrm{mod}\ p).$$

We write (2) : $$(p - 3)!\,(p - 2) - 1 \equiv 0 (\mathrm{mod}\ p).$$

$$(p - 3)!\,(p - 2) - 1 = p(p - 3)! - [2(p - 3)! + 1]$$

Consequently

(3) $$2(p - 3)! + 1 \equiv 0 (\mathrm{mod}\ p)$$

We write (3): $$2(p - 4)!\,(p - 3) + 1 \equiv 0 (\mathrm{mod}\ p)$$

$$2(p - 4)!\,(p - 3) + 1 = 2p(p - 4)! - [2 \cdot 3(p - 4)! - 1]$$

Consequently

(4) $$6(p - 4)! - 1 \equiv 0 (\mathrm{mod}\ p).$$

Now let us write the four congruences this way:

$$\left.\begin{array}{ll} (1a) & 0!(p-1)!+1 \\ (2a) & 1!(p-2)!-1 \\ (3a) & 2!(p-3)!+1 \\ (4a) & 3!(p-4)!-1 \end{array}\right\} \equiv 0 \pmod{p}.$$

This pattern suggests an algorithm. If this algorithm is true, (5a) would read

$$4!(p-5)!+1 \equiv 0 \pmod{p}.$$

We could proceed until the first number under the factorial x would be equal to the second number $[p-(x+1)]$, so that, if x would be even,

$$(6a) \qquad (x!)^2+1 \equiv 0 \pmod{p}.$$

If x would be odd

$$(7a) \qquad (x!)^2-1 \equiv 0 \pmod{p}.$$

Now we can show by induction that indeed the above algorithm is true. Let us assume that

$$(A) \qquad r![p-(r+1)]!+1 \equiv 0 \pmod{p}.$$

Then

$$r![p-(r+1)]!+1 = r![p-(r+2)]![p-(r+1)]+1$$
$$= r!\{p[p-(r+2)]!-(r+1)[p-(r+2)]!\}+1$$
$$= pr![p-(r+2)]!-\{(r+1)![p-(r+2)]!-1\}$$

Consequently

$$(B) \qquad (r+1)![p-(r+2)]!-1 \equiv 0 \pmod{p}.$$

$$(r+1)![p-(r+2)]!-1 = (r+1)![p-(r+3)]![p-(r+2)]-1$$
$$= (r+1)!\{p[p-(r+3)]!-[p-(r+3)]!(r+2)\}-1$$
$$= p(r+1)![p-(r+3)]!-\{(r+2)![p-(r+3)]!+1\}$$

Consequently

$$(C) \qquad (r+2)![p-(r+3)]!+1 \equiv 0 \pmod{p}.$$

The assumption of A corresponds to the true statement of (1a), B corresponds to (2a) and (4a), C to (1a) and (3a). So the algorithm is true.

II.

In order to look closer into the algorithm let us substitute n for $(p-1)$.

Except for $p = 2$, n is even, since p as a supposed prime number must be odd.

Let r be a whole number $r < n$. We split n into r and $n - r$. r and $n - r$ are, in any case, either both even or both odd. If they are even we have the congruence

$$r!(n-r)! + 1 \equiv 0 (\mathrm{mod}\ p) \tag{8}$$

If they are odd, we have the congruence

$$r!(n-r)! - 1 \equiv 0 (\mathrm{mod}\ \ ). \tag{9}$$

As an illustration we give p the value of 7, $n = 6$ and r the values from 0 to 3.

$$\begin{array}{ll}
(1a) & 0!\ 6! + 1 = 721 = 7 \cdot 103 \\
(2a) & 1!\ 5! - 1 = 119 = 7 \cdot 17 \\
(3a) & 2!\ 4! + 1 = 49 = 7 \cdot 7 \\
(4a) & 3!\ 3! - 1 = 35 = 7 \cdot 5
\end{array}$$

By each of these four calculations we have shown that 7 is prime. (4a) corresponds to (7a) and gives in the form $(3!)^2 - 1 \equiv 0 (\mathrm{mod}\ p)$ the shortest calculation. So we may, for practical purposes replace the Wilson formula (1) by (6a) and (7a). E.g., to find out whether 13 is prime, it is not necessary to figure $12! + 1$, but just $(6!)^2 + 1$.

$$6! = 720, \quad 720^2 = 518{,}400.$$

$$518{,}401 = 13 \cdot 39877.$$

But in case of (7a) we have still a further simplification.

$$[(n/2)!]^2 - 1 = [(n/2)! + 1][(n/2)! - 1]$$

So if p is prime, either $(n/2)! + 1$ or $(n/2)! - 1$ contains p as factor. In the case of $p = 7$, $n/2 = 3$, so that (7a) can be applied. $3! = 6$, $6 + 1 = 7$. Let

$$p = 11, \quad n/2 = 5, \quad 5! = 120, \quad 121 = 11 \cdot 11.$$

$$p = 23, \quad n/2 = 11, \quad 11! = 39916800,$$

$$39916799 = 23 \cdot 1735513$$

III

Finally let us have a look at $p = 2$. Here the splitting of $n = 1$ is possible only in one way: 0 and 1. Here $n = n - r$ is odd and r is even. But the algorithm finds a satisfying solution. In this particular case we have the particular congruence

$$0! \cdot 1! \pm 1 \equiv 0 \pmod 2).$$

New York

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

292. *Proposed by Eugenio Calabi and Chih-yi Wang, University of Minnesota.*

Find the area of the region in the real xy plane such that

$$|\sinh x \sinh y| < 1.$$

293. *Proposed by Raphael T. Coffman, Richland, Washington.*

Given a line of unit length, construct geometrically a line of length $(1 + 1/n)^n$, where n is an integer.

294. *Proposed by N. A. Court, University of Oklahoma.*

The nine-point centers of the four triangles formed by four concyclic points taken three at a time lie on a circle.

295. *Proposed by B. F. Cron, Underwater Sound Laboratory, New London, Connecticut.*

Let $f(x)$ be a function of x such that $\int_{-\infty}^{\infty} f(x)dx = 1$.

Prove that $\lim_{\Delta x \to 0} \sum f(x)\Delta x \log f(x)\Delta x$ diverges.

296. *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Prove that the following equalities

$$385 + 439 + 547 = 367 + 475 + 529$$

$$385^2 + 439^2 + 547^2 = 367^2 + 475^2 + 529^2$$

are true not only when the six distinct 3-digit numbers are considered to belong to the decimal system of numeration, but also when they are regarded as belonging to any system or scale of numerical notation with any base greater than ten.

297. *Proposed by Dewey Duncan, East Los Angeles Junior College.*

Given a point, a line and a circle in a plane. Construct an equilateral triangle having a vertex on each of them. Determine the criterion for the existence of such a triangle.

298. *Proposed by Huseyin Demir, Kandilli, Bolgesi, Turkey.*

Let $y = f(x)$ be a curve with the following properties

a) $f(x) = f(-x)$

b) $f'(x) > 0$ for $x > 0$

c) $f''(x) > 0$

Determine the weight per unit length $w(x)$ at the point (x,y) such that when the curve is suspended under gravity by any two points on it, the curve will keep its original shape.

SOLUTIONS

Late Solutions

259. *B. Keshava R. Pai, Lingraj College, Belgaum, India.*

269. *A.K. Rajagopal, Lingraj College. Belgaum, India; H.M. Gandhi, Lingraj College, Belgaum, India*

Errata

Problem 280 page 45 should read: If the cevian AD of the acute triangle ABC is the arithmetic, (geometric), [harmonic] mean of the sides b and c of the triangle, show that $\sin \delta$, δ being the acute angle between the cevian and a, is the harmonic, (geometric), [arithmetic] mean between $\sin B$ and $\sin C$.

Problem 283 page 46. The upper limit on the third summation symbol should read a_{n-1}.

Atomistic Chaos

271. [May 1956] *Proposed by C.W. Trigg, Los Angeles City College.*

In the scale of 8, the addition $ATOM = BOMB = CHAOS$ utilizes a $BOMB = 1$ (mod 7) and produces a $CHAOS$ which is a permutation of consecutive digits. Each letter uniquely represents a digit. Convert the letters into digits.

Solution by Herbert R. Leifer, Pittsburg, Pennsylvania. Arranging the addition as follows:

$$\begin{array}{r} A\ T\ O\ M \\ \underline{B\ O\ M\ B} \\ C\ H\ A\ O\ S \end{array}$$

it is immediately obvious that $C = 1$ and $M = 7$ and that the digits in *CHAOS* are either 0 through 4 or 1 through 5. Use of the 0 through 4 gives two letters the same value and thus results in no solution. Using 1 through 5 for *CHAOS*, B, t are 0 or 6. Since from the addition $M + B = S$, $B \neq 0$, $B = 6$, $T = 0$. Since $BOMB \equiv 1 \pmod 7$ or $6\phi 76 \equiv 1 \pmod 7$, ϕ is easily found to be 3. By substituting the now known values in the original

$$\begin{array}{r} A\,0\,3\,7 \\ 6\,3\,7\,6 \\ \hline 1\,H\,A\,3\,S \end{array}$$

The remaining letters are easily converted into digits

$$\begin{array}{r} 4\,0\,3\,7 \\ 6\,3\,7\,6 \\ \hline 1\,2\,4\,3\,5 \end{array}$$

Also solved by Monte Derham, San Francisco, California; Edgar Karst, IBM Corporation, Endicott, New York; Joseph D.E. Konhauser, State College, Pennsylvania; W. Moser, Unvversity of Saskatchewan; Frank Saunders, Coker College, Hartsville, South Carolina; Howard Schwartz, Student at Far Rockaway High School, Far Rockaway, New York; Chih-yi Wang, University of Minnesota and the proposer.

A Partition of Integers

272 [May 1956] *Proposed by M. Rumney, London, England.*

Given three different positive integers N_1, N_2, N_3. Find a partition of N_1 into three different positive integers a_{11}, a_{12} a_{13}, N_2 into three different positive integers a_{21}, a_{22}, a_{23} and N_3 into three different positive integers a_{31}, a_{32}, a_{33} such that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

Solution by Chih-yi Wang, University of Minnesota. Choose $a_{11} = a$, $a_{12} = b$, $a_{13} = N_1 - a - b$ such that they are distinct, choose $a_{21} = pa$, $a_{22} = pb$, $a_{23} = N_2 - pa - pb$, $p \geq 1$ such that they are distinct and then choose $a_{31} = qa$, $a_{32} = qb$, $a_{33} = N_3 - qa - qb$, $q \geq 1$ such that they are distinct. Clearly they satisfy the requirements.

Collinear Points

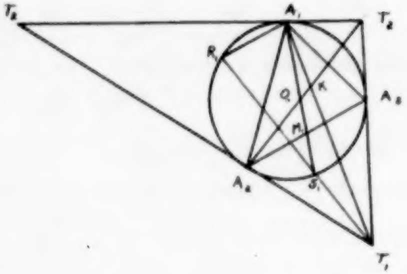
272. [May 1956] *Proposed by N.A. Court, University of Oklahoma.*

The points of intersection of the tangents to the circumcircle of a triangle drawn at the ends of one side is collinear with the two points which that circle marks on the median issued from the opposite vertex and on the parallel through that vertex to the side considered.

I. *Solution by J. W. Clawson, Collegeville, Pennsylvania.* It is well known that a median such as $A_1M_1S_1$ and the corresponding symmedian line A_1T_1 are isogonally conjugate, i.e. $\measuredangle A_2A_1T_1 = \measuredangle S_1A_1A_3$. Now, by symmetry about the line OM_1T_1, $\measuredangle T_1R_1A_1 = \measuredangle R_1A_1T_1 = \measuredangle R_1A_1A_2 + \measuredangle A_2A_1T_1 = \measuredangle A_3A_2A_1 + \measuredangle A_2A_1T_1$.

Again, $\measuredangle S_1R_1A_1 = \measuredangle S_1A_2A_1 = \measuredangle S_1A_2A_3 + \measuredangle A_3A_2A_1 = \measuredangle S_1A_1A_3 + \measuredangle A_3A_2A_1$.

Hence, $\measuredangle T_1R_1A_1 = \measuredangle S_1R_1A_1$, and R_1, S_1, T_1 are collinear.



II. *Solution by Huseyin Demir, Kandilli, Bolgesi, Turkey.* Let the median and exmedian relative to the vertex A intersect the circumcircle at E, F respectively, and let K_a be the intersection of the tangents at the other vertices B, C. From the harmonic ratios

$$A\ (B,\ C,\ E,\ F) = (AB,\ AC,\ AE,\ AF) = -1$$

we get a harmonic quadrangle $BECF$. This proves the diagonal EF contains K_a.

III. *Solution by Maimouna Edy, Hull, P.Q., Canada.* To generalize the problem, let ABC be a triangle inscribed in a given conic. Let

M and N be two points of the conic such that the lines CM and CN are harmonic conjugates with respect to CA and CB. The correspondence between M and N is then an involution on the conic and hence the line MN passes through a fixed point D, the Frégier point of the considered involution. A and B are double points of the involution, therefore D is the intersection of the tangents at A and B. Since the parallel through C to AB and the median through C are harmonic conjugates with respect to CA and CB, the desired conclusion is clear.

Also solved by Huseyin Demir, (two additional solutions); G.W. Courter, Baton Rouge, Louisiana; Joseph D.E. Konhauser, State College, Pennsylvania; W. Moser, University of Saskatchewan; Michael S. Pascual, Siena College, New York; A.K. Srinivasan and A.K.Rajagopal (jointly), Lingraj College, Belgaum, India; and the proposer (two solutions).

An Odd Game

274. [May 1956] *Proposed by Monte Dernham, San Francisco, California.*

Tom and Jerry together have m dollars. Tom gives Jerry as much money as Jerry already has. Then Jerry gives back to Tom as much as Tom has left, whereupon Tom gives Jerry as much as Jerry then has, and so on. After exactly n such transfers of cash, each of which serves to double the recipient's then existing pile, one of the two participants finds he has nothing left. (1) Which one? (2) How much did each have at the start? (3) What restriction on m is necessary and sufficient to justify the assumption that each started with an integral number of dollars?

Solution by William Moser, University of Saskatchewan. Suppose Tom starts with t dollars and Jarry with $m - t$ dollars. After n transfers: Tom has $T(2k) = 2^{2k}t - 2/3\,(4^k-1)m$, Jerry has $J(2k) = \dfrac{2\cdot 4^k + 1}{3}\cdot m - 2^{2k}t$ when $n = 2k$; Tom has $T(2k+1) = 2^{2k+1}\,t = \dfrac{4^{k+1}-1}{3}\cdot m$, Jerry has $J(2k+1) = \dfrac{4^{k+1}+2}{3}\cdot m - 2^{2k+1}\,t$ when $n = 2k+1$. Since $T(2k) = 2T(2k-1)$ and $J(2k+1) = 2J(2k)$ it follows that, if the game ends when $n = 2k$, then Tom has all the money and $t = \dfrac{2\cdot 4^k + 1}{3\cdot 2^{2k}}\cdot m$; and if the game ends when $n = 2k+1$ then Jerry has all the money and

$t = \dfrac{4^{k+1} - 1}{3 \cdot 2^{2k+1}} \cdot m$. Clearly a necessary and sufficient condition that t be integral is that m be integral and divisible by an integral power of 2.

Also solved by Maimouna Edy, Hull, P.Q., Canada; Herbert R. Leifer, Pittsburg, Pennsylvania and the proposer.

Combinations of Primes

275. [may 1956] *Proposed by L. Carlitz, Duke University.*

I. Let the prime $p = 3m + 1$. Show that

$$\sum_{r=0}^{m-1} \binom{3r+1}{r} \left(\frac{1}{9}\right)^r \equiv \left\{ \begin{array}{ll} -3 & (m \equiv 1 (\text{mod } 3)) \\ 3 & (m \equiv -1 (\text{mod } 3)) \\ 0 & (m \equiv 0 (\text{mod } 3)) \end{array} \right\} \pmod p$$

$$\sum_{r=0}^{m-1} \binom{3r+1}{r} \left(\frac{2}{27}\right)^r = 0 \pmod p.$$

II. Let the prime $p = 4m + 1$. Show that

$$\sum_{r=0}^{m-1} \binom{4r+1}{2r} \left(\frac{1}{16}\right)^r \equiv (-1)^{m-1} \pmod p.$$

Solution by the proposer. We shall use the formula (see for example Todhunter's Trignometry)

$$\frac{\sinh 2mx}{\sinh 2x} = \sum_{r=0}^{m-1} \binom{m+r}{2r+1} 2 \sinh x^{2r}$$

This may be rewritten as

$$(1) \qquad \frac{u^{2m} - u^{-2m}}{u^2 - u^{-2}} = \sum_{r=0}^{m-1} \binom{m+r}{2r+1} (u - u^{-1})^{2r}.$$

I. $p = 3m + 1$. Then $m \equiv -1/3 \pmod p$.

Since

$$\binom{-\frac{1}{3} + r}{2r+1} = -\frac{1}{3} \; \frac{\frac{2}{3}\,\frac{5}{3} \cdots \frac{3r-1}{3}\left(-\frac{4}{3}\right)\cdot\left(-\frac{7}{3}\right) \cdots \left(-\frac{3r+1}{3}\right)}{(2r+1)!} = -\frac{(-1)^r}{3^{3r+1}}\binom{3r+1}{r}.$$

(1) implies

$$(2) \qquad -3\,\frac{u^{2m} - u^{-2}}{u^2 - u^{-2}} = \sum_{0}^{m-1} (-1)^r \binom{3r+1}{r} \frac{(u - u^{-1})^{2r}}{3^{3r}} \pmod p.$$

Let $u = w$, where $w^2 + w + 1 = 0$. Then $w^3 = 1$ and $(w - w^{-1})^2 = w^2 - 2 + w^{-2} = -3$, so that the right member of (2) reduces to

$$\sum_0^{m-1} \binom{3r+1}{r} \frac{1}{3^{2r}}$$

As for the left member, we have

$$\frac{w^{2m} - w^{-2m}}{w^2 - w^{-2}} = \frac{w^{2m} - w^m}{w^2 - w} = \begin{cases} 1 & m \equiv 1 \pmod 3 \\ -1 & m \equiv -1 \pmod 3 \\ 0 & m \equiv 0 \pmod 3. \end{cases}$$

This proves the first result.

II Next take $u = \dfrac{1+i}{\sqrt{2}}$, $i = \sqrt{-1}$; these symbols are at any rate meaningful in the $GF(p^4)$. Then $u^2 = i$ and $(u - u^{-1})^2 = -2$. Thus the right member of (2) reduces to

$$\sum_0^{m-1} \binom{3r+1}{r} \left(\frac{2}{27}\right)^r.$$

As for the left member, we have (since m is even)

$$\frac{u^{2m} - u^{-2m}}{u^2 - u^{-2}} = \frac{i^m - i^{-m}}{2i} = \frac{(-1)^{m/2} - (-1)^{m/2}}{2i} = 0.$$

This proves the second result.

III. $p = 4m + 1$. Then $m \equiv -\frac14 \pmod p$. Since

$$\binom{-\frac14 + r}{2r+1} = \frac14 \; \frac{\frac34 \, \frac74 \cdots \frac{4r-1}{4}\left(-\frac54\right)\left(-\frac94\right)\cdots\left(-\frac{4r+1}{4}\right)}{(2r+1)!} = -\frac{(-1)^r}{4^{3r+1}}\binom{4r+1}{2r}$$

(1) becomes

$$-4\,\frac{u^{2m} - u^{-2m}}{u^2 - u^{-2}} \equiv \sum_0^{m-1} (-1)^r \binom{4r+1}{2r} \frac{(u - u^{-1})^{2r}}{4^{3r}} \pmod p. \tag{3}$$

Now take $u = i$, which in this case is certainly an integer $\pmod p$. Then the right member of (3) reduces to

$$\sum_{0}^{m-1} \binom{4r+1}{2r} + \frac{1}{16^r}$$

We rewrite the left member as $-4(u^{2m-2} + u^{2m-6} + \ldots + u^{-2m+2})$, which becomes $-4mi^{2m-2} = -4m(-1)^{m-1} = (-1)^{m-1}$.

This completes the proof of the third result.

Simultaneous Equations

276. [May 1956] *Proposed by P.H. Yearout, Portland Oregon.*

Solve simultaneously

$$x + y + z = 2 \qquad (1)$$
$$x^2 + y^2 + z^2 = 14 \qquad (2)$$
$$x^4 + y^4 + z^4 = 98 \qquad (3)$$

I. *Solution by Dermott A. Breault, Pittsburg, Pennsylvania.* Assuming integral solution, we see that in (3) no one of the unknowns can be > 3. Since the resulting sum of fourth powers (all of which must be positive) would exceed 98.

Accordingly we note that both (2) and (3) are satisfied by $x = \pm 3$, $y = \pm 2$, $z = \pm 1$. (4) However, of these eight possible solutions (4), only 3, -2, 1, is a solution of (1)

$\therefore \quad x = 3$

$y = -2$ or any permutation thereof is the set

$z = 1$ of solutions to the system (1)(2)(3).

II. *Solution by Hazel Schoonmaker Wilson, Jacksonville State College, Alabama.* Squaring (1) and subtracting from (2), gives

$$xy + xz + yz = -5 \qquad (4)$$

Squaring (2) and subtracting from (3), gives

$$x^2y^2 + x^2z^2 + y^2z^2 = 49 \qquad (5)$$

Squaring (4) and subtracting from (5), gives

$$2x^2yz + 2xy^2z + 2xyz^2 = -24$$
$$xyz(x + y + z) = -12$$
$$xyz = -6 \qquad (6)$$

From (1), (4), and (6) we know that x, y, z are the roots of the cubic equation

$$x^3 - 2x^2 - 5x + 6 = 0.$$

These roots are easily found, by synthetic division, to be 1, -2, and 3.

Also solved by Huseyin Demir, Kandilli, Bolgesi, Turkey; Monte Dernham, San Francisco, California (partially); Maimouna Edy, Hull, P.Q.,Canada; H.M. Gandhi, Lingraj College, Belgaum, India; M.N. Gopalan, Mysore, India; Gaines B. Lang, University of Florida; C.N. Mills, Augustana College, South Dakota; Washin Ng, San Francisco, California; A.K. Rajagopal, Lingraj College, Belgaum, India (two solutions); C.D. Smith, University of Alabama; Chih-yi Wang, University of Minnesota and the proposer.

A Game of Chance

277. [May 1956] *Proposed by J.M. Howell, Los Angeles City College.*

In a game, A has probability 2/3 of winning \$1, probability 5/24 of losing \$1, and probability 1/8 of losing \$2. A starts with \$10 and his opponent has unlimited resources. What is the probability of A's wealth not decreasing to \$1 or less? We assume that A must stop if he has only \$1, since he could not pay a \$2 loss.

Solution by the proposer. Let P_x be the probability of A winning all the money, if he starts with x dollars, and opponent has $n - x$ dollars.

Then, $P_x = (2/3)P_{x+1} + (5/24)P_{x-1} + (1/8)P_{x-2}$ for $x = 2, 3, \ldots, n-1$

Boundary conditions are: $P_0 = 0$, $P_1 = 0$, $P_n = 1$.

Assume $P_x = r^x$, then $r^x = (2/3)r^{x+1} + (5/24)r^{x-1} + (1/8)r^{x-2}$.

Multiplying by r^{2-x}, $r^2 = (2/3)r^3 + (5/24)r + (1/8)$

or, $16r^3 - 24r^2 + 5r + 3 = (r-1)(4r-3)(4r+1) = 0$. $r = 1, \frac{3}{4}, -\frac{1}{4}$.

Then $(1)^x$, $(\frac{3}{4})^x$, and $(-\frac{1}{4})^x$ are independent solutions of above difference equation, and $P_x = c_1 + c_2(\frac{3}{4})^x + c_3(-\frac{1}{4})^x$ *is the general solution.*

Evaluating constants by using boundary conditions, we get:

$$P_0 = 0 = c_1 + c_2 + c_3 \qquad P_1 = 0 = c_1 + (\tfrac{3}{4})c_2 - (\tfrac{1}{4})c_3$$

$$P_\infty = \lim P_n = 1 = c_1 \qquad c_1 = 1,\ c_2 = -5/4,\ c_3 = \tfrac{1}{4}\ .$$

Then if $x = 10$, and approximate solution is:

$$P_x = 1 - (5/4)(3/4)^{10} + (1/4)(-1/4)^{10} \doteq .92961$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 187. Prove $\sqrt{\dfrac{ab + bc + ca}{3}} \geq \sqrt[3]{abc}$ where $a, b, c \geq 0$.

[*Submitted by M. S. Klamkin*]

Q 188. At what times must the hands of a clock be interchanged in order to obtain new correct time? [*Submitted by Huseyin Demir*].

Q 189. Evaluate $\displaystyle\sum_{r+1}^{n+1} \sum_{s+1}^{m} \sin[(2r - 2s - 1)\theta]$

[*Submitted by M.S. Klamkin*]

Q 190. Solve

$$y = x - \frac{x^2}{2}$$

$$x = \tfrac{1}{2} + \frac{y^2}{2}$$

[*Submitted by M.S. Klamkin*]

ANSWERS

A 187. Let $ab + bc + ca = s$ then the maximum value of $(ab)(bc)(ca)$ occurs when $ab = bc = ca$ or $a = b = c$.

Thus $\sqrt{\dfrac{ab + bc + ca}{3}} \geq \sqrt[3]{abc}$ for all $a, b, c \geq 0$.

A 188. Let p denote the number of hours and α the fraction of an hour at the time T. When the hands are interchanged we obtain new time t', the corresponding numbers being p', α' (We may suppose $p' \geq p$). At t the angle in hours $p' + \alpha'$ of minute hand is 12 times α:

$$p' + \alpha' = 12, \quad 0 \leq p' < 12, \quad \alpha' < 1.$$

$$p + \alpha = 12\alpha', \quad 0 \leq p < 12, \quad \alpha < 1.$$

We have

$$\alpha = (12p' + p)/143$$

$$\alpha' = (p' + 12p)/143 .$$

Hence, given any two positive integers p, p' less than 12 we get α and α', and therefore the required times.

A 189. Since

$$(2r - 2s - 1)\,\theta + [2(n + 2 - r) - 2(n - 1 + s) - 1]\,\theta = 0$$

it follows that the given sum is zero.

A 190. $y = \frac{1}{2} + \frac{y^2}{2} - \frac{x^2}{2}$ so $(y - 1)^2 = x^2$ or $x = \pm (y - 1)$

Thus $1 + y^2 = 2(1 - y)$ and $1 + y^2 = 2(y - 1)$ so we have

$$y = -1 \pm \sqrt{2} \qquad \text{and} \qquad y = 1 \pm i\sqrt{2}$$

$$x = 2 \mp \sqrt{2} \qquad\qquad x = i\sqrt{2}$$

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 26. If $\tan 3A \tan 2B \tan C = \tan 3A - \tan 2B - \tan C$ prove that $3A = 2B + C$. [*Submitted by Miss S.S. Shirali*]

T 27. A person traveling eastward at a rate of 3 miles per hour finds that the wind appears to blow directly from the north. On doubling his speed it appears to come from the north east. What was the wind velocity? [*submitted by M.S. Klamkin*]

T 28. Find a function of a variable such that if the variable varies in a Geometric Progression the function varies in Arithmetic Progression. [*Submitted by B. Keshava R. Pai*]

SOLUTIONS

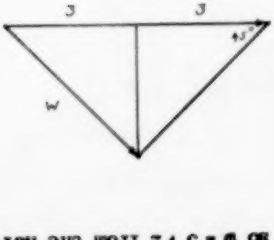
S 26. Since

$$\tan(3A - 2B - C) = \frac{\tan 3A - \tan 2B - \tan C - \tan 3A \tan 2B \tan C}{1 - [\tan 2B \tan C - \tan 3A \tan 2B - \tan 3A \tan C]}$$

if we substitute in the given condition we have tan

$$\tan(3A - 2B - C) = 0 \quad \text{or} \quad 3A = 2B + C.$$

S 27. The vector triangle is as follows, so $w = 3\sqrt{2}$ from the north west.



S 28. Solution: $x = \log_a y$.